Singaporean students' mathematical thinking in problem solving and problem posing: An exploratory study

Article *in* International Journal of Mathematical Education In Science & Technology · January 2003

DOI: 10.1080/00207390310001595401

1 author:

Jinfa Cai
University of Delaware

INT. J. MATH. EDUC. SCI. TECHNOL., 2003
VOL. 34, NO. 5, 719–737

Singaporean students' mathematical thinking in problem solving and problem posing: an exploratory study

JINFA CAI
Department of Mathematical Sciences, University of Delaware,
Newark, DE 19716, USA
E-mail: jcai@math.udel.edu

(*Received 10 October 2002*)

This study explored Singaporean fourth, fifth, and sixth grade students' mathematical thinking in problem solving and problem posing. The results of this study showed that the majority of Singaporean fourth, fifth, and sixth graders are able to select appropriate solution strategies to solve these problems, and choose appropriate solution representations to clearly communicate their solution processes. Most Singaporean students are able to pose problems beyond the initial figures in the pattern. The results of this study also showed that across the four tasks, as the grade level advances, a higher percentage of students in that grade level show evidence of having correct answers. Surprisingly, the overall statistically significant differences across the three grade levels are mainly due to statistically significant differences between fourth and fifth grade students. Between fifth and sixth grade students, there are no statistically significant differences in most of the analyses. Compared to the findings concerning US and Chinese students' mathematical thinking, Singaporean students seem to be much more similar to Chinese students than to US students.

1. Introduction

Recently published results from cross-national comparative studies of the teaching and learning of mathematics, such as the Third International Mathematics and Science Study (TIMSS) and the Third International Mathematics and Science Study-Repeat (TIMSS-R), have received considerable attention from the educational research community as well as from the general public. Based on the results of the TIMSS [1], Singaporean fourth and eighth grade students ranked top of several nations in all mathematical content areas. Results from TIMSS-R also showed that Singapore compiled the highest average achievement scores in mathematics. Because of the high achievement scores of the Singaporean students, educators and politicians called for effort to learn from Singapore. Singaporean mathematics textbooks have been widely sold around the world. Some school districts in the USA have adopted Singaporean school textbooks.

Because teaching is a cultural activity [2], we may not be able to directly transplant a successful teaching practice from one culture into another. On the other hand, the examination of a successful instructional practices in one culture may provide knowledge and experience to handle the issues and challenges in mathematics education of the another culture. International comparative studies provide unique opportunities to understand the current state of students' learning

International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390310001595401

and indicate ways to enhance their future learning [2,3]. What are the factors and the circumstances that have led to Singaporean students' high achievement in mathematics? Because of the complexity of interpreting cross-national performance differences, there is no simple answer to the question. One may argue that it is the changes made in Singapore's educational system that led to this significant improvement in mathematics performance in the past decade [4]. It may be also due to the coherent integration of Eastern and Western cultures in Singaporean society [5]. However, the purpose of this paper is not to try to examine cultural factors and educational systems, and then to understand why Singaporean students ranked top in large-scale international assessments. Rather, the purpose of this paper is to explore Singaporean students' mathematical thinking in problem solving and problem posing.

In order to improve students' learning, it is necessary to understand the developmental status of their thinking and reasoning. The more information teachers obtain about what students know and think, the more opportunities they create for student success in the classroom [6,7]. It is important to look beyond the rankings in international assessments [3]. An overemphasis on the rankings of countries by school achievements sidetracks the search for what we can learn from cross-national studies to improve students' learning [8]. The position taken in this paper is that the main goal of educational research, including comparative research in education, is to improve the learning opportunities for all students. The purpose, then, of international studies is to provide information about how we can improve students' learning in mathematics instruction. In particular, the purpose of this exploratory study is to provide some information about Singaporean students' mathematical thinking and reasoning and discuss the findings from a cross-national comparative perspective.

2. Theoretical considerations for examining students' thinking

The theoretical considerations for examining students' mathematical thinking and reasoning are based on recent advances of performance assessment and cognitive psychology. There is an increased use of open-ended problems in national and international assessments [9–12]. Student responses to open-ended tasks are analyzed to capture the high-level thinking and reasoning processes. The level of students' thinking and reasoning can be captured through examination of their use of solution strategies, display of mathematical domain knowledge, representation of solution processes, justification of mathematical reasoning, and posing of new problems based on a problem situation. These cognitive aspects are identified as important and significant in cognitive psychology in general [13–17] and mathematical problem solving in particular [18–22].

2.1. *Solution strategies*

Individual differences in solving mathematical problems can sometimes be understood in terms of differences in the uses of different strategies. Proficiency in solving mathematical problems is dependent on the acquisition, selection, and application of both domain-specific strategies and general cognitive strategies [16,23]. Thus, competence in using appropriate problem-solving strategies reflects students' degrees of performance proficiency in mathematics. This implies that assessment tasks should reveal the various strategies that students employ. In addition, students' problem-solving strategies become more effective over time. Therefore, both the examination of the strategies that students apply and the success of those applications can provide information regarding students' developmental status of mathematical thinking and reasoning.

2.2. *Mathematical justification*

Researchers have been interested in all aspects of reasoning processes, including collecting evidence, making inferences, and justifying conclusions [24,25]. National Council of Teachers of Mathematics [20] suggests that the study of mathematics should emphasize reasoning so that students can justify their answers and solution processes as well as make and evaluate mathematical conjectures and arguments. Mathematical justification is related to communication. With this increasing awareness of the importance of communications in instruction, it is imperative that mathematical communication be an important dimension in assessing students' mathematics proficiency. For example, the Mathematical Sciences Education Board [26] proposed requiring communication in task development and evaluation of students' responses. Through examining students' mathematical justifications of their solutions, much can be learned about students' reasoning and communication skills. Students' mathematical justifications should be judged in terms of their soundness: (1) the justification providing support is acceptable or correct and (2) the justification providing support is complete or incomplete [24,25].

2.3. *Solution representations*

In solving a problem, a problem solver first needs to establish a representation of the problem [27]. The problem representation includes the initial state of the problem (the 'givens') and the goal of the problem. As would be expected, the development of the problem representation largely influences the success of the problem solution. Experts tend to represent problems based on concepts and principles whereas novices tend to recognize only the surface features of the problems. Thus, representations used by experts facilitated their problem solving [13]. Solution representations are the external representations of students' solution processes, which reflect their mathematical thinking. Examination of the solution representation reveals the ways in which students process a problem and reflects the ways students communicate their mathematical ideas and thinking processes.

2.4. *Mathematical problem posing*

Problem posing is one of the key components of mathematical exploration. In scientific inquiry, formulating a problem well is often a more significant task than finding solutions to the problem [28]. Moreover, focusing on how students pose problems helps illuminate what can be learned from studying how students solve problems, and vice versa [20,22,29,30]. For example, an examination of students' problem solving can help us understand the solution strategies and representations students use to solve given problems. We can then examine their mathematical problem posing to see if there are analogous patterns in the ways they present the problems they generate. Therefore, problem posing focuses on the investigation of students' thinking from different perspectives.

3. Methods

3.1. *Subjects*

A total of 155 fourth graders, 167 fifth graders, and 150 sixth graders from four Singaporean elementary schools participated in the study. Male and female students are about evenly distributed in each grade. The four selected schools represent different levels according to students' overall academic performance. The first school is among the top four in Singapore and the second school is among the top 20. The third school is about average, and the last school is below average. The selection of different levels of students is an effort to have a representative sample. For the purpose of this study, results were reported in an aggregated manner.

3.2. *Tasks and data coding*

Four tasks (shown in the Appendix) were used in this study. The examination of Singaporean textbooks and interviews with Singaporean teachers showed that their students learned all of the concepts involved in these tasks. These tasks are mathematically rich and are embedded in different content areas and contexts and allow for examining Singaporean students' thinking from various perspectives. The Hats Average Task requires students to find the missing number when three of four numbers and the average of the four numbers are presented in a picture. To solve this problem, students cannot directly apply the averaging algorithm in the traditional 'add them up then divide' way. Since knowing the 'average' is an uncommon situation, a correct solution requires the flexible and reversible application of the algorithm, and students may use various solution strategies and representations to solve the problem. The Odd Number Pattern Problem is embedded in a party context in which an odd number of guests enter the party on each ring. The number of guests who enter the house at ring n can be represented as $2n-1$, but students need not use the algebraic representation in order to solve the problem. However, the problem does require students to use regularities to extend the pattern and to effectively communicate these regularities by explaining how they got their answers. This task allows for an examination of students' generalization skills. To solve the Pizza Ratio Task, students need to set up a ratio for the number of pizzas and the number of people and then compare the fraction representations. Students need to determine if each girl or each boy gets more pizza. To solve this problem, students may use numerical or pictorial representations. The Problem-Posing Task allows for the examination of the generative aspects of mathematical thinking.

Responses to each of the problem-solving tasks (i.e. Hats Average Task, Pizza Ratio Task, and Odd Number Pattern Task) were coded for three aspects: correctness of answer; type of solution strategy or justification; and form of solution representation. Details about solution strategies and representations are described in the results section. Previous studies have suggested that this type of analysis is an appropriate way to capture the mathematical thinking and reasoning involved in problem solving [9]. In other words, examination of the strategies that students apply and their success in applying them can provide information regarding the robustness of students' mathematical thinking and reasoning. This information is complemented by examining the kinds of representations students choose to use, which in turn, reflect the ways in which students process a problem and communicate their mathematical ideas and thought processes. Responses to the problem-posing task were coded to capture the kinds of problems students generated. The detailed categories of posed problems are described in the results section.

The data were coded by two research assistants. To ensure inter-coder reliability, the two research assistants independently coded the same 15 booklets in each grade level. The inter-coder agreements were 88–94% for coding correctness of answers, solution strategies, and solution representations in problem-solving tasks. The inter-rater agreements were 82–90% for coding responses to the problem-posing task coder.

4. Results

4.1. *Results from the hats average problem*

Correctness of numerical answers and error analysis. Across the three grade levels, the percentages of students who had the correct answers increased from 73% of the fourth graders to 93% of fifth graders and to 95% of sixth graders $[\chi^2(2, N=472)=39.66, p<0.001]$. In particular, a larger percentage of the fifth graders than fourth graders provided the correct answer $(z=4.78, p<0.01)$. Although a slightly larger percentage of the sixth graders than of the fifth graders had the correct answer for the problem, the difference between the two was not statistically significant.

Error analysis of the 42 fourth grade students, 11 fifth grade students, and 8 sixth grade students who had incorrect answers was conducted using a category scheme developed and used in [31] to code Chinese and US students' errors in solving a similar Averaging Problem. Besides the minor computation errors of Singaporean fifth and sixth grade students, the most frequently committed error is students' incorrect use of the averaging algorithm (26% of 42 fourth graders, 18% of 11 fifth graders, and 25% of 8 sixth graders). Below is an example of the error of incorrect use of averaging algorithm:

> The student added the number of hats sold in week 1 (9), week 2 (3), and week 3 (6), then divided the sum by 3, and got 6. However, the average was 7. Therefore, the student added 3 to the sum of the numbers of hats sold in the first three weeks, then divided it by 3, and got 7, and then gave the answer 3.

A considerable number of the fourth, fifth, and sixth grade students (14% of 42 fourth graders, 18% out of 11 fifth graders, 25% of the 8 sixth graders) just found the total number of hats sold in four weeks (28) as the answer. Nearly 15% (out of 42) of the fourth grade students simply added up some numerals given in the problem and recorded the sum in the answer space. For example, one student counted the number of hats sold in week 1 (9), week 2 (3), and week 3 (6). Then the student wrote down all numerals in the problem, such as 1 in 'Week 1', 2 in 'Week 2', 3 in 'Week 3', 4 in 'Week 4', and 4 and 7 in 'How many hats must Angela sell in Week 4 so that the average number of hats sold is 7?' Finally, the student added all the numbers that appeared in the problem $(9+3+6+1+2+3+4+4+7=39)$ and gave 39 as the answer. One Singaporean fifth grader and one sixth grader made the same error as well.

Solution strategies and representations. In order to understand the differences in this mathematical performance among the three grade levels, students' solution strategies and representations in the problem solving process were also analyzed. The vast majority of the fourth (81%), fifth (94%), and sixth (96%) grade students correctly used the averaging formula to solve the problem (e.g. $7 \times 4 - (9 + 3 + 6) = 10$). A few of the students in each grade used either the leveling strategy or the guess-and-check strategy. Using the leveling strategy, the student viewed the average (7) as a leveling basis to 'line up' the numbers of hats sold in the week 1, 2, and 3. Since 9 hats were sold in week 1, it has two extra hats. Since 3 hats were sold in week 2, 4 additional hats are needed in order to line up the average. Since 6 hats were sold in week 3, it needs 1 additional hat to line up the average. In order to line up the average number of hats sold over four weeks, 10 hats should be sold in week 4. Using the guess-and-check strategy, the student first chose a number for week 4, and then checked to see if the average of the numbers of hats sold for the four weeks was 7. If the average was not 7, then the student chose another number for the week 4 and checked again, until the average was 7. About 20% of the fourth graders' explanations were not clear enough to detect their solution strategies. Percentages of fifth and sixth grade students with unclear explanations were very small (about 3%).

The examination of representations provides another perspective of students' thinking in problem solving. Each of the students' responses was analyzed for its representations. Three categories were used to evaluate and classify representations in the student's explanation: verbal (primarily written words), visual (a picture or drawing), or symbolic (mathematical notations). The majority of the students in each grade level used mathematical notations (88% for fourth graders, 95% for fifth graders, and 99% for sixth graders), but as the grade level advances, the higher percentage of students in that grade level show evidence of using symbolic representations [$\chi^2(2, N = 472) = 16.60, p < 0.01$). There is no statistically significant difference between fifth and sixth graders, but a larger percentage of sixth graders than fourth graders used symbolic representations ($z = 3.77, p < 0.01$). Symbolic representations can involve either arithmetic (e.g. $7 \times 4 - (9 + 3 + 6) = 10$) or algebraic (e.g. $(9 + 3 + 6 + x) = 7 \times 4$, and then solve for x) solutions. For those students who used symbolic representations, only one fourth grader, three fifth graders, and four sixth graders used algebraic representations to solve the problem.

4.2. *Results from the odd number pattern problem*

Correctness of answers. Students are required to answer three questions to solve the Odd Number Pattern Problem. Table 1 shows the percentages of students who obtained correct answers for questions A and C and the percentage of students who provided descriptions of the rule which can be used to find the number of guests entering on each ring. For the question to find the number of guests who entered on the 10th ring, there is a significant difference across the three grade levels [$\chi^2(2, N = 472) = 11.04$, $p < 0.01$]. That overall difference across the three grade levels is due to the fact that a larger percentage of fifth graders (93%) than fourth graders got the correct answer ($z = 2.65$, $p < 0.01$). When students were asked to write a rule or describe in words how to find the number of guests who entered on each ring of the doorbell, the majority of the students were able to come up with a rule. However, across the three grade levels, there is a significant difference [$\chi^2(2, N = 472) = 10.07$, $p < 0.01$]. Similarly, the overall difference across the three grade levels is mainly due to the fact that a larger percentage of fifth graders (92%) than of fourth graders describe a rule ($z = 2.17$, $p < 0.05$).

	Percentage of students		
	Fourth grade ($n = 155$)	Fifth grade ($n = 167$)	Sixth grade ($n = 150$)
Number of guests on the 10th ring	83	93	93
Described a rule	85	92	95
Ring number for 99 guests	33	45	63

Table 1. Percentages of students correctly answering three questions for the odd number pattern problem.

In answering which ring of the doorbell determines that 99 guests have entered, Chi-square tests showed significant differences across the three grade levels [χ^2 (2, $N = 472$) = 28.73, $p < 0.001$]. In particular, a larger percentage of fifth grade students (45%) than of fourth grade students (33%) provided the correct answer ($z = 2.20, p < 0.05$). Similarly, a larger percentage of sixth grade students (63%) than fifth grade students correctly found the ring number when 99 guests entered ($z = 3.16, p < 0.01$).

Solution strategies. The kinds of strategies students used to find the number of guests on the 10th ring and the ring number when 99 guests entered were examined. The kinds of rules students described were also examined. Cai [9] classified the strategies used to find the number of guests entering on the 10th ring into two categories: concrete and abstract. Using a concrete strategy, students made a table or a list or noticed that each time the doorbell rang two more guests entered than on the previous ring, and actually added 2s sequentially to find an answer to describe a rule. Using an abstract strategy, some students noticed that the number of guests who entered on a particular ring of the doorbell equalled two times that ring number minus one (i.e. $y = 2n - 1$, where y represents the number of guests and n represents the ring number). Others noticed that the number of guests who entered on a particular ring equalled the ring number plus the ring number minus one (i.e. $y = n + (n - 1)$, where y represents the number of guests and n represents the ring number). Then using the generalized rule, students found the number of guests who entered on the 10th ring.

The kinds of rules students described can also be classified as concrete and abstract. However, in addition to the concrete and abstract strategies, a third category is needed to classify the strategies used to find the ring number at which 99 guests had entered: the computation strategy. Using a computation strategy, students showed a number of steps of computation to yield a correct answer, but it is not completely clear why they performed these computations. Below are two examples:

Example 1. $99 - 9 = 90$. $90 \div 2 = 45$. $45 + 5 = 50$.

Example 2. $99 - 7 = 92$. $92 \div 2 = 46$. $46 + 4 = 50$.

Table 2 shows the percentages of students in each grade level with concrete and abstract strategies. The majority of the students in each grade level used concrete strategies to find the number of guests entering on the 10th ring. 'Making a list' is the dominant concrete strategy for students in each grade level (56, 84, and 83% for fourth graders, fifth graders, and sixth graders, respectively). Only a small

	Percentage of students		
	Fourth grade ($n=155$)	Fifth grade ($n=167$)	Sixth grade ($n=150$)
Strategies to find the number of guests on 10th ring			
Concrete	91	92	92
Abstract	1	6	5
No strategies	8	2	3
The kinds of rules described			
Concrete	84	86	85
Abstract	1	6	10
No strategies	15	8	5
The ring number for 99 guests			
Concrete	43	46	33
Abstract	12	16	37
Computation	3	6	8
No strategies	42	32	22

Table 2. Percentages of students with concrete and abstract strategies for answering each question of the odd number pattern problem.

proportion of the students in each grade level used abstract strategies to find the number of guests who entered on the 10th ring. Similarly, while the vast majority of the students in each grade level described rules in a concrete way, only a small percentage of them in each grade described rules in an abstract way. However, the percentages of students in each grade level who used abstract strategies to find the ring number when 99 guests had entered increased significantly when compared to the percentages of students in each grade level who used abstract strategies to find the number of guests entered on the 10th ring. For example, 12% of the fourth graders used abstract strategies to find the ring number for 99 guests, but only 1% of students in the same grade used abstract strategies to find the number of guests entering on the 10th ring ($z=3.84$, $p<0.01$). Similarly, for the fifth graders, the percentage of students who used an abstract strategy to find the ring number for 99 guests (16%) is greater than that of students finding the number of guests on the 10th ring (6%) ($z=2.96, p<0.01$). For the sixth graders, there is a dramatic increase (from 5 to 37%) in the percentage of students who used abstract strategies to find the ring number for 99 guests compared to those finding the number of guests on the 10th ring ($z=6.76, p<0.01$).

The finding that more students in each grade level used abstract strategies to find the ring number for 99 guests than to find the number of guests on the 10th ring might be related to the nature of the questions. People usually try to solve problems using the most comfortable and viable strategy [16]. However, the most comfortable strategy is not necessarily the most sophisticated and efficient strategy. Making a list or continuing to add by 2s is a viable strategy to quickly find the number of guests on the 10th ring. Therefore, it is not surprising that over 90% of the students in each grade level used such a concrete strategy. On the other hand, although making a list or continuing to add by 2s to find the ring number for 99 guests is a viable strategy, it is very inefficient and time consuming. For some students, an abstract strategy involving 'undoing' (e.g. since 2 times the ring minus 1 is the number of guests entering on each ring number, $99 + 1 = 100$. $100 \div 2 = 50$) seems to be a more accessible way to find the ring number for 99 guests.

Across the three grades, the students' grade level is associated with their use of abstract strategies. The higher the grade level, the more likely students are to use abstract strategies. This is particularly true for finding the ring number when 99 guests enter: about 12% of fourth graders used abstract strategies, but the percentage increased to 16% for fifth graders and 37% for sixth graders [χ^2 $(2, N = 472) = 32.81, p < 0.001$].

4.3. *Results from the pizza ratio problem*

Levels and kinds of justifications. This task required students to justify whether each girl gets the same amount of pizza as each boy, and if not, who gets more. Students' justifications were classified into four levels:

(1) *Complete and convincing argument*: If there were six girls, each girl and each boy would have the same. But you have 8 girls, so each girl gets less than each boy.
(2) *Vague or incomplete argument*: Boys get bigger slices and girls get smaller slices. So boys get more than girls.
(3) *Incorrect or incomprehensible arguments*: You can cut girls' pizza into eight pieces and cut the boys' pizza into three pieces. You get more pieces for girls than boys, so each girl gets more. and
(4) *No argument*.

Table 3 shows percentages of students' different levels of justification. There is a significant difference across the three grade levels [χ^2 $(2, N = 472) = 46.11$, $p < 0.01$]. The overall difference across the three grade levels is due to the fact that a larger percentage of fifth graders (92%) than fourth graders provided complete and convincing arguments to justify that each girl gets a different amount from each boy and that each boy gets more than each girl ($z = 5.62$, $p < 0.001$).

Cai [9] identified eight types of justifications when he examined Chinese and US students' thinking as they solved a similar Pizza Ratio Problem. Singaporean students used three of the eight types of justifications, shown below. For those students in each grade level who provided complete and correct justifications, the vast majority of them used type 1 justification (89% of the fourth graders, 97% of the fifth graders, and 96% of the sixth graders).

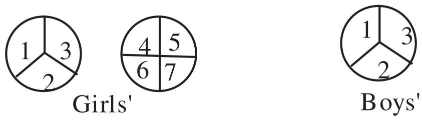
	Percentage of students		
	Fourth grade ($n = 155$)	Fifth grade ($n = 167$)	Sixth grade ($n = 150$)
Complete and convincing argument	65	92	96
Vague or incomplete argument	10	4	1
Incorrect or incomprehensible argument	17	2	3
No argument	8	2	0

Table 3. Percentages of students with various justifications for the pizza ratio problem.

Type 1: $2 \div 7 = 2/7$. $1 \div 3 = 1/3$. 1/3 is greater than 2/7 by transforming them into common fractions ($1/3 = 7/21$ and $2/7 = 6/21$. $7/21 - 6/21 = 1/21$) or decimals ($1/3 = 0.33$ and $2/7 = 0.29$. $0.33 - 0.29 = 0.04$), so each boy gets more than each girl.

Type 2: Seven girls get two pizzas, and three boys get one pizza. The girls have twice as many pizzas as the boys. But the number of girls is more than twice that of the boys. So each boy gets more than each girl.

Type 3: Three girls share one pizza, and the remaining four share one pizza. Each piece that each of the remaining four girls get is smaller than the pieces the boys get. So the boys get more.



Representations. One of the distinctive features in their justifications was that an increasingly greater percentage of the students used numerical symbols as their grade level advances. Figure 1 shows the percentage distributions of students' representations used in their justifications. In particular, 54% of the fourth grade students used numerical symbols, which increased to 66% for the fifth graders and 89% for the sixth graders [$\chi^2(2, N = 472) = 46.00, p < 0.001$]. In contrast, percentages of students who used visual drawings decreased from 17% for fourth graders to 14% for fifth graders and to 5% for sixth graders [$\chi^2(2, N = 472) = 13.21, p < 0.01$]. Figure 1 reveals the change in students' four representations across the three grade levels.

4.4. *Results from the problem posing task*

The problems students posed were coded into three levels. The problems were first classified into mathematical problems, nonmathematical or irrelevant problems, and no responses. Then each mathematical problem was classified into an extension problem or a nonextension problem. An extension problem refers to a problem questioning the pattern beyond the three given figures (How many white dots are there in the 30th figure?). A nonextension problem refers to a problem questioning the given figures in the pattern (How many black dots in the second figure?). Third, a nonextension problem was coded as questioning the number of black and/or white dots in a figure, number of black and/or white dots in two or more figures, comparing white and black dots in one or more figures, or drawing a

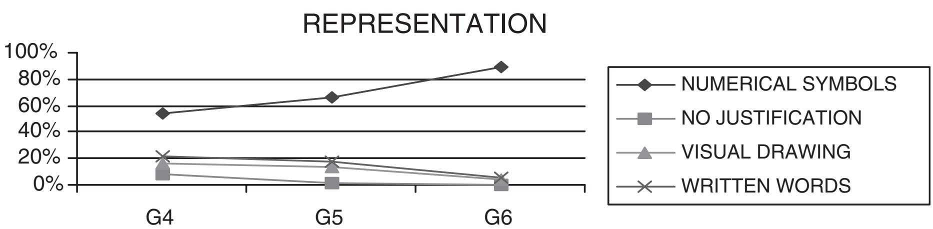


Figure 1. Percentage distributions of students' representations in each grade level for the pizza ratio problem.

figure in the pattern/shape of a figure. Besides the categories described for classifying a nonextension problem, an extension problem can be a rule-based general problem, or a rule-based specific problem. A rule-based general problem is usually vague and cannot be answered in a specific way (e.g. What is the pattern in these figures?). A rule-based specific problem has the specifics that one needs to know to solve the question (e.g. How do black dots in each figure increase?). Table 4 includes these coding categories and percentages of students in each category of the problems.

Mathematical and nonmathematical problems. Each student was asked to generate three problems. Table 5 shows the percentages of students generating various problems. About three-quarters of the fourth graders generated mathematical problems (including extension and nonextension mathematical problems); the percentage significantly increased to over 90% of fifth graders and over 95% of sixth graders in each of the three responses [χ^2 (2, $N = 472) = 36.11 - 43.53$, $p < 0.001$]. In each grade level, the percentages of the mathematical problems are about the same in each of the three responses. Nearly 15% of fourth graders did not respond to the problem-posing task, the percentage decreased to 2% for fifth graders and 1% for sixth graders. About 10% of fourth graders posed nonmathematical or irrelevant problems in each of the three responses. Here is an example of a nonmathematical problem: 'What materials made these circles?' 'Are they true circles' is an example of an irrelevant problem. For sixth graders, only a very few of them generated nonmathematical or irrelevant problems.

Extension and nonextension mathematical problems. The mathematical problems posed by the students were of particular interest and they were subjected to further analyses. Students tended to pose more extension problems as their grade level advanced. In particular, for their first problem, only 25% of fourth graders generated extension problems, but the percentage significantly increased to 55% for fifth graders and to 62% for sixth graders [$\chi^2 (2, N = 472) = 47.52, p < 0.001$]. There are similar increases of extension problems from the fourth grade to the fifth grade and to the sixth grade for their posed second problem [$\chi^2 (2, N = 472) = 53.35, p < 0.001$] and third problem [$\chi^2 (2, N = 472) = 55.37, p < 0.001$]. The overall difference across the three grade levels is mainly due to the difference between fourth and fifth grade students in their first posed problems. No statistically significant difference was found between fifth and sixth grade students regarding the percentages of extension problems. Within each grade level, more extension problems from their first posed problem to the second posed problem and to the third posed problem were generated by the fourth grade students [$\chi^2 (2, N = 472) = 17.88, p < 0.001$], the fifth grade students [$\chi^2 (2, N = 472) = 19.36, p < 0.001$] and by the sixth grade students [$\chi^2 (2, N = 472) = 18.37, p < 0.001$].

Table 4 shows the kinds of extension problems posed by students. Across the three grade levels, the most frequently generated problems are the following: (1) asking for the number of dots in a figure; (2) asking to draw a figure/shape of a figure; and (3) asking for rule based general problems. Within these three kinds of problems, the rule based general problems tend to be the most popular problems generated by the fourth and fifth grade students, while problems asking for the number of dots in one figure are the most popular problems for sixth grade students. There is a clear increase of the extension problems related to asking for

		Fourth grade ($n=155$)			Fifth grade ($n=167$)			Sixth grade ($n=150$)		
		*P*1	*P*2	*P*3	*P*1	*P*2	*P*3	*P*1	*P*2	*P*3
Extension math problem	Dots in one figure	4	5	12	13	22	32	30	35	46
	Dots in more than one	1	3	3	0	2	2	1	3	3
	Comparing number of dots	0	1	3	1	4	7	3	9	18
	Draw a figure/shape of a figure	8	9	14	15	8	12	7	7	4
	Rule-based/general	10	14	15	25	28	25	15	9	13
	Rule-based/specific	2	1	1	1	2	0	6	8	0
	Total % of extension problems	*25*	*33*	*48*	*55*	*66*	*78*	*62*	*71*	*84*
Nonextension math problem	Dots in one figure	10	5	2	10	4	3	6	6	2
	Dots in more than one	9	13	10	10	4	2	7	7	3
	Comparing number of dots	28	21	12	16	17	9	21	12	6
	Draw a figure/shape of a figure	1	1	1	0	2	2	1	1	0
	Total % of nonextension problems	*48*	*40*	*25*	*36*	*27*	*16*	*35*	*26*	*11*
No mathematical or irrelevant problems		*9*	*8*	*11*	*8*	*3*	*3*	*2*	*1*	*2*
No responses		*18*	*19*	*16*	*1*	*4*	*3*	*1*	*2*	*3*

Table 4. Percentage distributions of posed various problems.

	Percentage of students		
	Fourth grade (n=155)	Fifth grade (n=167)	Sixth grade (n=150)
Mathematical problems			
Response 1	75	92	98
Response 2	72	91	97
Response 3	73	90	95
Nonmathematical problems/no responses			
Response 1	25	8	2
Response 2	28	9	3
Response 3	27	10	5

Table 5. Percentages of mathematical and nonmathematical problems/no responses.

the number of dots in one figure and comparing the number of dots in figures across the three grade levels. For example, in their third generated problem, only 12% of fourth graders asked for the number of dots in a figure, but the percentage increased to 32% for fifth graders, and 46% for sixth graders [$\chi^2(2, N=472) = 41.86, p < 0.001$]. Similarly, in their third generated problem, only 3% of fourth graders asked for comparing the number of dots in figures, but this increased to 7% for fifth graders, and 18% for sixth graders [$\chi^2(2, N=472) = 21.08, p < 0.001$]. Only a small proportion of students in each grade level generated rule-based specific problems and problems comparing numbers of dots in figures. In contrast, nonextension problems involving comparing numbers of dots in figures are the most frequently posed problems across the three grade levels. In addition to that, nonextension problems asking for the number of dots in more than one figure is also very popular. In fact, a considerable number of students in each grade asked for the total number of white or black dots in the three given figures.

Progression of difficulty levels of posed problems. Each student was asked to generate three problems: an easy problem, a moderately difficult problem, and a difficult problem. Previous studies showed some evidence that students might have thought about the solutions to the problems they posed. If that is the case, we should actually see the progression of difficult levels of the problems students posed in this study. Since an extension problem is usually more difficult than a nonextension problem, the fact that students in each grade level posed more extension problems in their third response than in their first and second responses indicates a progression of difficulty levels of problems posed from their first response to the second and third responses. In addition, comparisons of difficulty levels of problems generated by each student were made using the following assumptions and criteria:

(1) Only those students who generated at least two mathematical problems were included in this analysis.
(2) An extension problem is more difficult than a nonextension problem.
(3) For extension problems, a rule-based specific problem is more difficult than other problems.

(4) A problem involving comparing number of dots in figures is more difficult than asking for the number of dots in one of the figures involved.
(5) A problem involving combining the number of dots in figures is more difficult than asking for the number of dots in one of the figures involved.
(6) A problem asking a student to draw a figure in the pattern is more difficult than the one for asking a student to find the number of dots in the figure.
(7) A problem involving a later figure in the pattern is more difficult than a problem involving an earlier figure.

In total, 107 fourth graders (69%), 149 fifth graders (89%), and 142 sixth graders (95%) are included in the analysis. Table 6 shows the percentage distributions of the students in each category examining the progression of difficulty levels of posed problems. If we combine students in each grade level with full progressive problems as $P1 < P2 < P3$ and partial progressive problems as $P1 < P3$ and $P2 < P3$ or $P1 < P2$ and $P1 < P3$, nearly half or slightly over half of the students in each grade generated partial or full progressive problems. Across the three grade levels, there is a similar percentage (about 20%) of students having at least one pair of the problems whose earlier posed problem is more difficult than the later posed problem. In each grade level, about 25% of the students' problems could not be compared in terms of their difficulty levels. This result is largely due to students' posing rule-based general problems, which are usually unsolvable.

Across the three grade levels, a higher percentage of the students having full progressive difficulty levels of problems as $P1 < P2 < P3$ [$\chi^2(2, N = 398) = 8.21$, $p < 0.05$], but there is no significant difference between fourth and fifth graders as well as between fifth and sixth graders. There are 23% of fourth grade students having partial progressive problems as $P1 < P3$ and $P2 < P3$ or $P1 < P2$ and $P1 < P3$, which is significantly larger than the percentages for fifth graders ($z = 2.55, p < 0.05$) and sixth graders ($z = 3.25, p < 0.01$).

5. Discussion

This study has explored Singaporean fourth, fifth, and sixth grade students' mathematical thinking in problem solving and problem posing. In particular, student responses to three problem-solving tasks were analyzed in terms of the types of solution strategies, justifications, and representations. The results of this study showed that the majority of Singaporean fourth, fifth, and sixth graders are

	Percentage of students		
	Fourth grade ($n = 107$)	Fifth grade ($n = 149$)	Sixth grade ($n = 142$)
$P1 < P2 < P3$	30	40	48
$P1 < P3$ and $P2 < P3$, or $P1 < P2$ and $P1 < P3$	23	9	11
Having at least one of the following: $P1 > P2$, $P2 > P3$, $P1 > P3$	21	23	19
Unable to compare difficulty level	26	28	22

Table 6. Percentages of students in each category of examining progression of difficulty level of posed problems.

able to select appropriate solution strategies to solve these problems, and choose appropriate solution representations to clearly communicate their solution processes. In addition, this study also analyzed student responses to a problem-posing task in terms of the kinds of problems students generated. Most Singaporean students are able to pose problems beyond the initial figures in the pattern (extension problems).

The results of this study also showed that across the four tasks, as the grade level advances, a higher percentage of the students in that grade level show evidence of having correct answers. Surprisingly, the overall statistically significant differences across the three grade levels are mainly due to the statistically significant differences between the fourth and fifth grade students. Between the fifth and sixth grade students, there are no statistically significant differences in most of the analyses. There are similar results related to the problem-posing task. Even though there are overall differences across the three grade levels in terms of the kind of mathematical problems posed, there is no statistically significant difference between the fifth and sixth grade students. One possible interpretation of this finding is that, after the fifth grade level, the percentage of students who got the correct answer for each of the problem-solving tasks reached a high and stable level. For example, over 90% of fifth and sixth grade students obtained the correct answer for the Hats Average Problem. For those fifth and sixth grade students who have not reached a high level, there was also a statistically significant difference. For example, for the third question of the Odd Number Pattern Problem (asking for the ring number when 99 guests entered), the sixth graders have a significantly higher success rate than do the fifth graders. This result also appears to support the above interpretation. However, future studies are needed to understand fully about the developmental difference for the Singaporean students.

Similar tasks have been used to examine US and Chinese students' mathematical thinking [9,29], therefore, we can discuss Singaporean students' mathematical thinking from an international comparative perspective. There are some similarities among Singaporean, Chinese, and US students. For example, for the Hats Average Problem, like the Chinese and US students, the majority of Singaporean students use the averaging formula to solve the problem. Like the Chinese and US students, Singaporean students' most common error when solving the Hats Average Problem is their incorrect use of the averaging algorithm. However, across the four tasks, Singaporean students seem to be much more like Chinese students than like US students. For example, an earlier study [9] showed that none of the Chinese sixth graders used any pictorial representations to solve the Hats Average Problem, but nearly 10% of the US students used pictorial representations. Similar to Chinese students, only very few Singaporean students used pictorial representations to solve the Hats Average Problem.

For solving the Pizza Ratio Problem, like the Chinese students, Singaporean students used one dominant strategy to justify why each boy gets more pizza than each girl [$2 \div 7 = 2/7$. $1 \div 3 = 1/3$. $1/3$ is greater than $2/7$ by transforming them into common fractions ($1/3 = 7/21$ and $2/7 = 6/21$. $7/21 - 6/21 = 1/21$) or decimals ($1/3 = 0.33$ and $2/7 = 0.29$. $0.33 - 0.29 = 0.04$), so each boy gets more than each girl]. However, US students used quite a few different visual drawing strategies. For the Odd Number Pattern Problem, like the Chinese students, a considerable number of Singaporean students used abstract strategies, but the US students rarely used abstract strategies.

For the Problem Posing Task, Chinese, Singaporean and US students are alike in terms of the kinds of problems posed. The only noticeable difference is related to the progression of the three posed problems. Like Chinese students, Singaporean students showed a clear progression in the posed problems. The most common progression begins with a problem that involves the given figure in the statement of the task. This is often followed by a problem that attempts to generate or make use of a pattern. The last problem in the sequence may involve further application of the pattern, or it may consist of a continued attempt to analyse the given information to establish the pattern. This progression of posed problems is not evident for the US students [29].

The only noticeable difference between Chinese and Singaporean students is related to the use of algebraic symbolic representations in solving the Hats Average Problem. In the earlier study [9], it was found that nearly 20% of the Chinese sixth graders used algebraic symbolic representations to solve the problem, but in this study, only a few Singaporeans used algebraic symbolic representations. In this regard, Singaporean students are similar to US sixth grade students who rarely set up equations to solve the Hats Average Problem.

Acknowledgments

The preparation of this paper was, in part, supported by a grant from the National Science Foundation (ESI-0114768). The author wishes to acknowledge valuable assistance with data collection by Mrs Khoo Chung Teik. Thanks also go to Yi Long, Eric Sisofo, and Tao Wang for data coding and analysis.

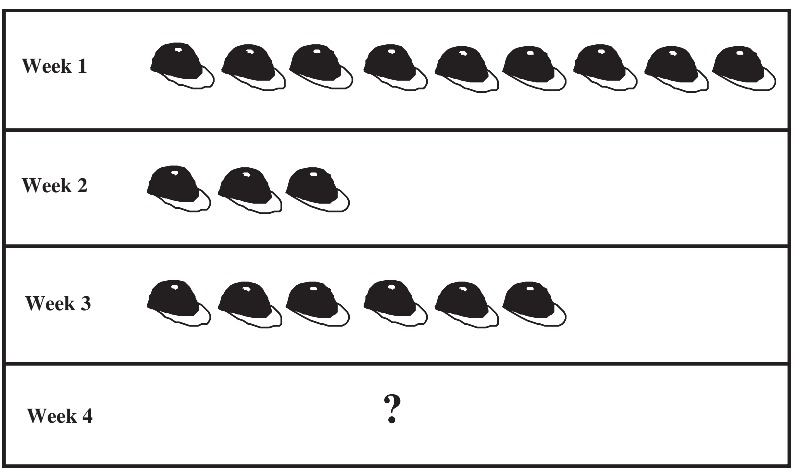
Appendix: Tasks

The Hats Averaging Problem

Angela is selling hats for the Mathematics Club. This picture shows the number of hats Angela sold during the first three weeks.

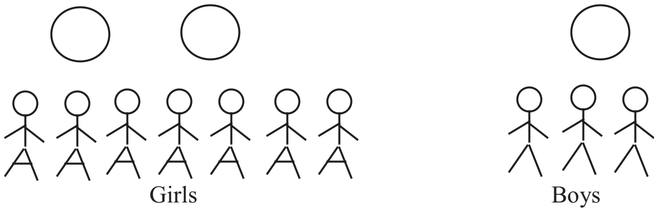
How many hats must Angela sell in Week 4 so that the *average* number of hats sold is 7?

Show how you found your answer.



The Pizza Ratio Problem

Here are some children and pizzas. 7 girls share 2 pizzas equally and 3 boys share 1 pizza equally.



A. Does each girl get the same amount as each boy?
Explain or show how you found your answer.
B. If each girl does not get the same amount as each boy, who gets more?
Explain or show how you found your answer.

The Odd Number Pattern Problem

Sally is having a party.

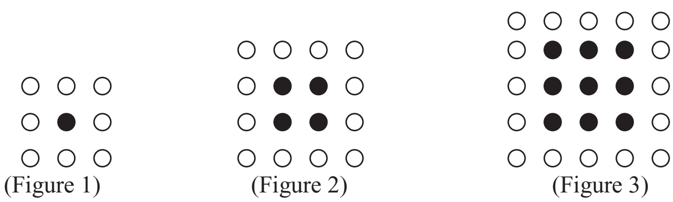
The first time the doorbell rings, 1 guest enters.
The second time the doorbell rings, 3 guests enter.
The third time the doorbell rings, 5 guests enter.
The fourth time the doorbell rings, 7 guests enter.

Keep going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

A. How many guests will enter on the 10th ring?
Explain or show how you found your answer.
B. In the space below, write a rule or describe in words how to find the number of guests that entered on each ring.
C. 99 guests entered on one of the rings. What ring was it?
Explain or show how you found your answer.

Problem Posing Task

Mr Su drew the following figures in a pattern, as shown below.



For his student's homework, he wanted to make up three problems BASED ON THE ABOVE SITUATION: an easy problem, a moderate problem, and a difficult problem. These problems can be solved using the information in the situation.

Help Mr Su make up three problems and write these problems in the space below.

The easy problem
The moderately difficult problem
The difficult problem

References

[1] Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., and Smith, T. S., 1996, *Mathematics achievement in the middle school years: IEA's Third International Mathematics and Science Study* (TIMSS, Boston College, MA: TIMSS International Study Center).

[2] Stigler, J. W., and Hiebert, J., 1999, *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom* (New York: The Free Press).

[3] Cai, J., 2001, *Phi Delta Kappa*, **82,** 400.

[4] Lim, S. K., 2002, In *Mathematics education for a knowledge-based era: Proceedings of International Congress of Mathematics Instruction's Second East Asian Regional Conference on Mathematics Education,* edited by D. Edge and B. H. Yeap, Volume 1: *Plenary and Regular Lectures*) (Association of Mathematics Educators and National Institute of Education in Singapore, May 27–31, 2002), pp. 29–37.

[5] Leung, F. K. S., 2002, Mathematics education in East Asian and the West: Does culture matter? Paper presented at the International Congress of Mathematics Instruction's Comparative Study Conference. The University of Hong Kong, Hong Kong, China. October 20–25, 2002.

[6] Darling-Hammond, L., 1994, *Harvard Educ. Rev.*, **64**(1), 5.

[7] Pellegrino, J. W., Chudowsky, N., and Glaser, R. (eds), 2001, *Knowing What Students Know: The Science and Design of Educational Assessment* (National Academy Press).

[8] Westbury, I., 1992, *Educ. Res.*, **21,** 18.

[9] Cai, J., 2000, *Math. Thinking Learning: Int. J.*, **2,** 309.

[10] Harmon, M., Smith, T. A., Martin, M. O. *et al.*, 1997, *Performance Assessment in IEA's Third International Mathematics and Science Study* (TIMSS, Chestnut Hill, MA, Boston College: TIMSS International Study Center).

[11] National Assessment Governing Board, 1994, *Mathematics Framework for the 1996 National Assessment of Educational Progress* (Washington, DC: National Assessment Governing Board).

[12] Mislevy, R. J., 1995, *Educ. Eval. Policy Anal.*, **17,** 419.

[13] Bransford, J. D., Brown, A. L., and Cocking, R. R., 2000, *How People Learn: Brain, Mind, Experience, and School,* expanded edition (Washington, DC: National Research Council).

[14] Magone, M., Cai, J., Silver, E. A., and Wang, N., 1994, *Int. J. Educ. Res.*, **21,** 317.

[15] Royer, J. M., Cisero, C. A., and Carlo, M. S, 1993, *Rev. Educ. Res.*, **63,** 201.

[16] Simon, H. A., 1989, *Models of Thought,* Vol. II (New Haven: Yale University Press).

[17] Snow, R. E., and Lohman, D. F., 1989, In *Educational Measurement,* 3rd edn, edited by R. L. Linn (New York: Macmillan), pp. 263–332.

[18] Grouws, D. A. (ed.), 1992, *Handbook of Research on Mathematics Teaching and Learning* (New York: Macmillan).

[19] Kilpatrick, J., 1987, In *Cognitive Science and Mathematics Education,* edited by A. H. Schoenfeld (Hillsdale, NJ: Erlbaum), pp. 123–147.

[20] National Council of Teachers of Mathematics, 2000, *Principles and Standards for School Mathematics* (Reston, VA: NCTM).

[21] Schoenfeld, A. H. (ed.), 1987, *Cognitive Science and Mathematics Education* (Hillsdale, NJ: Erlbaum).

[22] Silver, E. A., 1994, *For Learning Math.*, **14**(1), 19.

[23] Pressley, M., 1990, *Cognitive Strategy Instruction That Really Improves Children's Academic Performance* (New York: Brookline Books).
[24] English, L. D., 2000, *Mathematical Reasoning: Analogies, Metaphors, and Images* (Mahwah, NJ: Erlbaum).
[25] Voss, J. F., Perkins, D. N., and Segal, J. (eds), 1991, *Informal Reasoning and Education* (Hillsdale, NJ: Lawrence Erlbaum Associates).
[26] Mathematics Science Board, 1993, *A Conceptual Guide for Mathematics* (Washington, DC: National Academy Press).
[27] Newell, A., and Simon, H. A., 1972, *Human Problem Solving* (Englewood Cliffs, NJ: Prentice Hall).
[28] Einstein, A., and Infeld, L., 1938, *The Evolution of Physics* (New York: Simon & Schuster).
[29] Cai, J., and Huang, S., 2002, *J. Math. Behav.*, **21,** 401.
[30] English, L. D., 1997, *Educ. Studies Math.*, **34,** 183.
[31] Cai, J., 2000, *Int. J. Math. Educ. Sci. Technol.*, **31,** 839.