Numerical methods midterm project

I. We discussed in class several ways to estimate $\sqrt{2}$, some a one-shot estimation or bounds, others an iterative process that is all but guaranteed to get better with repetition. Still other methods exist where our target value is the convergent value of an infinite sequence, sum, product, or continued fraction. Broadly we can call these friendly neighbor methods, find the root methods, and limit as $n \to \infty$ methods, respectively

A. Match each function to its method:

$f(x) = \sqrt{x}$	Convergent value	Limit at infinity
$g(x) = x^2 - 2$	Linearization	
$h(x) = \dfrac{\sqrt{2x^2+1}}{x}$	Intermediate value theorem	Friendly Neighbor
	Newton's method	
$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$	Proportional reasoning	Find the root

B. Demonstrate your understanding by

1) showing how to find a fractional estimate of $\sqrt{3}$ accurate to within 1/16th using the intermediate value theorem. Write up your explanation here but use a spread sheet to do and show the computations.

2) showing how to find $\sqrt[3]{61}$ to 4 decimal places using the Newton's method. Write up your explanation here but use a spread sheet to do and show the computations.

C. One judgment of a good formula is one that can give you better and better fractional estimate of the irrational number we wish to find. This is a fancy way of restating a goal of only needing the basic four operations +,-,*,/. In either of the two radical computations above did you need more than a four-function computation? If so, could you, by choosing friendly neighbors, avoid this? Why or why not? Explain.

II. An underlying aim of this project, as I have stated often in class, is to understand how complicated values such as $e^{2/3}$ or $\ln\left(\frac{4}{3}\right)$ or $\tan(1)$ can be "computed" at all (note that by *computed* we actually mean estimated to predetermined degree of accuracy). We all take for granted, at this point in history, what an amazing devise a scientific calculator really is!

A. We begin our exploration by estimate $\tan(1)$ via linearization about the value $\frac{\pi}{3}$.

1) Write the equation of the tan line to $y = \tan x \;\; at \; x = \pi/3$

2) Write the exact form of your estimate of tan 1. Note the arithmetic required here is still largely limited to your basic four function calculator as long as you remember your trig facts. The exceptions are $\sqrt{3}$ and π

Use the result from I.B.1) for the former and use 22/7 for the latter to find a fractional estimate of $\tan 1$. Rhetorical question: Is this one numerical method inside another? Might we ever need even more nested methods?

E.c. Look up the history of 22/7 and pi. Write one thing you noticed and one thing you wonder about.

B. Next, we will explore what the most powerful of these methods can reveal about computing something like e to a fractional exponent. It's a rabbit hole so hang on.

To write a Newton's method formula, we must first find an equation with $e^{\frac{p}{q}}$ as a root.

Let $x = e^{\frac{p}{q}}$ where p and q are positive integers

$$\ln x = \frac{p}{q}$$

$$\ln x - \frac{p}{q} = 0$$

Using $f(x) = \ln x - \frac{p}{q}$ Newton's method formula is

$$x_{n+1} = x_n - \frac{\ln x_n - \frac{p}{q}}{\frac{1}{x_n}}$$

1) Why must we find a function with our target computation as a root?
2) Why wouldn't $f(x) = x - e^{\frac{p}{q}}$ be an equally good choice? Try it. what happens?
3) Simplify the Newton's method formula above. What computation in the formula cannot be done on a four-function calculator? Any ideas about how to "compute" (i.e. accurately estimate) that piece with only fractions? Try one. Does it work? Do you need yet more num methods for any of its parts?
4) One way to answer the last question come from looking ahead to the next course. In Chapter 7, You will learn that exponential functions are in fact defined in terms of natural logs which in turn are defined as the area under one of our much used curves $y = 1/x$
   a) Draw y=1/x
   b) Since we have run into $\ln 2$ before as the limit on $\frac{2^h-1}{h}$ $as\ h \to 0$ (do you remember where?), we will explore estimating that value by area. Mark the powers of 2 on the x-axis. All the areas between adjacent powers are equal to each other with the value: $\ln 2$. Is this surprising?
   c) Rectangular blocks of width one and heights determined by the curve can be used to over estimate and under estimate the true area. Using the blocks between 16 and 32, find an upper sum, a lower sum then average to estimate ln(2). Write up your explanation here but use a spread sheet to do and show the computations.
   d) Does the method get better using the blocks further right on the graph? Why or why not explain?
   e) Compare the simplicity of operations involved between this area method and our previous run in with ln(2) in the limit definition of $\frac{d}{dx}(2^x)$.
5) Return to step (3) with $p/q = 1/2$ and $x_o = 8/5$.
   Depending on simplification we get $x_1 = \frac{8}{5} - \frac{8}{5}\left(\ln\left(\frac{8}{5}\right) - \frac{1}{2}\right) = \frac{8}{5}\left(\frac{3}{2} - \ln\frac{8}{5}\right)$
   Replace $\ln\frac{8}{5}$ with an averaged area estimation using blocks between 20 and 32 and compute x_1. Repeat to find x_2. Write up your explanation here but use a spread sheet to do and show the computations.
6) Compute the absolute and relative error as compared with a scientific calculator's estimation of $e^{1/2}$.