Communication, Connections, and Conceptual Clarity

Dr. Lee Singleton

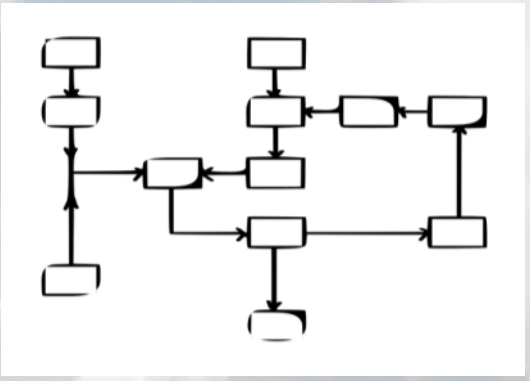
AMATYC Virtual Conference 2020

Friday, Nov. 20 – Hosted by Wiley

Collaborators

- Eric Davishahl
  - Engineering, WCC
  - PI of NSF Grant (WCC portion)
- Todd Haskell
  - Psychology, WWU
  - PI of NSF Grant (WWU portion)

Do Students Learn Concepts or Procedures?



$$Area = \int_a^b f(x) - g(x)dx$$

The integral is the area under a curve.

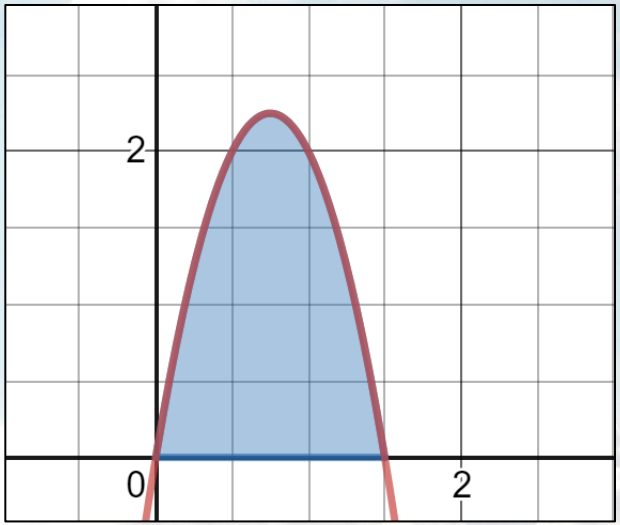
$$S.A. = \int_a^b 2\pi x\sqrt{1+\left(f'(x)\right)^2}\,dx$$

The center of mass of an object is in the middle of the object.

Integration is adding up a bunch of small things.

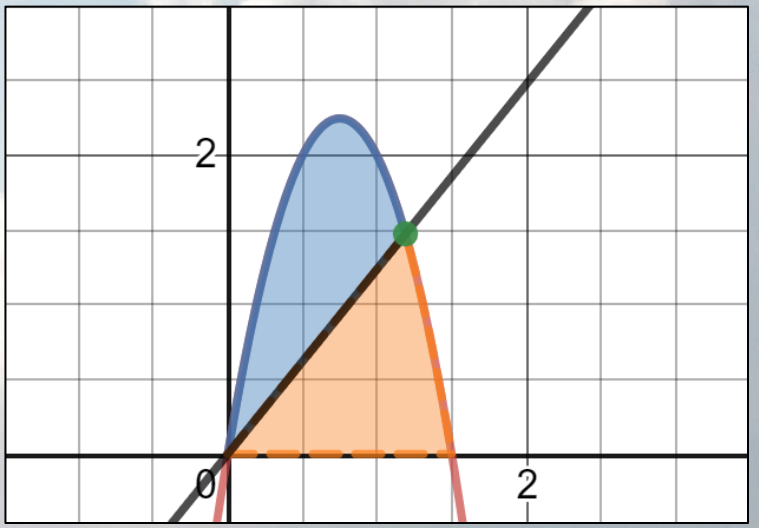
$$(\bar{x},\bar{y}) = \left(\int_a^b x\left(f(x)-g(x)\right)dx, \int_a^b \left(f(x)\right)^2 - \left(g(x)\right)^2\,dx\right)$$

Can You Show Me The Steps?



Find the area bounded by the parabola $y=6x-4x^2$ and the x-axis.

There is a line through the origin that divides the region bounded by the parabola $y=6x-4x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

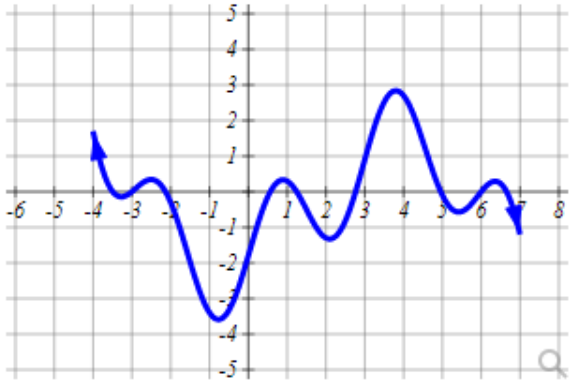


Concept Check Questions

MC+E Assessment Questions

- Multiple Choice + Explain Questions
- 1 pt for choice

- 4 pts for correct reasoning
- 2 pts for partially correct reasoning
- 0 pts for incorrect reasoning



The function $f(x)$ is shown in the graph above.

Put the following in order from smallest to largest:

A. $\int_0^3 f(x)dx$ B. $\int_{-2}^1 f(x)dx$ C. $\int_3^6 f(x)dx$ D. $\int_1^5 f(x)dx$

- ○ $A < B < C < D$
- ○ $A < B < D < C$
- ○ $B < A < C < D$
- ○ $B < A < D < C$

Sample Rubric for Concept Check Responses

Points	Justification
5	Correct choice selection with enough reasoning to justify the correct response.
4	Correct reasoning, but an incorrect selection from the choices.
3	Correct choice selection, but only partially correct reasoning to justify the response.
2	Incorrect choice selection, but partially correct reasoning in the explanation.
1	Correct choice selection, but little to no reasoning to justify the response.
0	Incorrect choice selection, and little to no justifiable reasoning.

Participant Poll

The function $f(x)$ is ***strictly decreasing*** on the entire interval $[a, b]$.

Which of the following is in order from smallest to largest?

- $R_5 < \int_a^b f(x)dx < L_5 < L_{10}$
- $L_5 < L_{10} < \int_a^b f(x)dx < R_5$
- $R_5 < \int_a^b f(x)dx < L_{10} < L_5$
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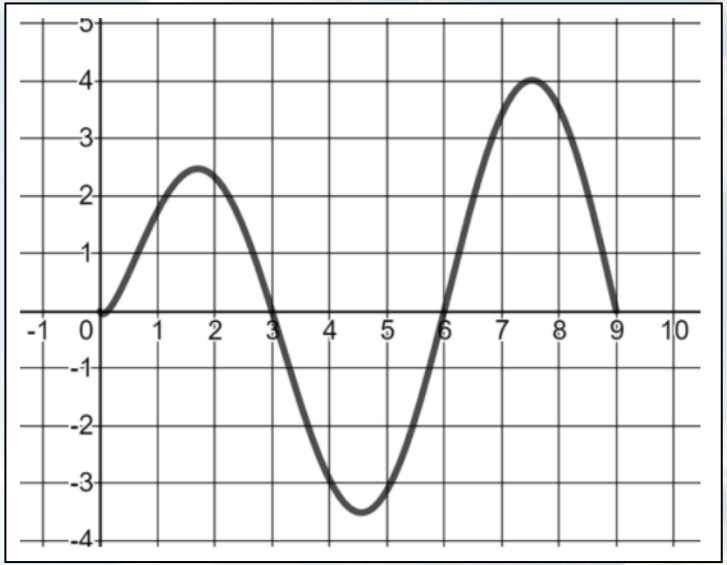
Correct Answer:

$$R_5 < \int_a^b f(x)\,dx < L_{10} < L_5$$

Participant Poll

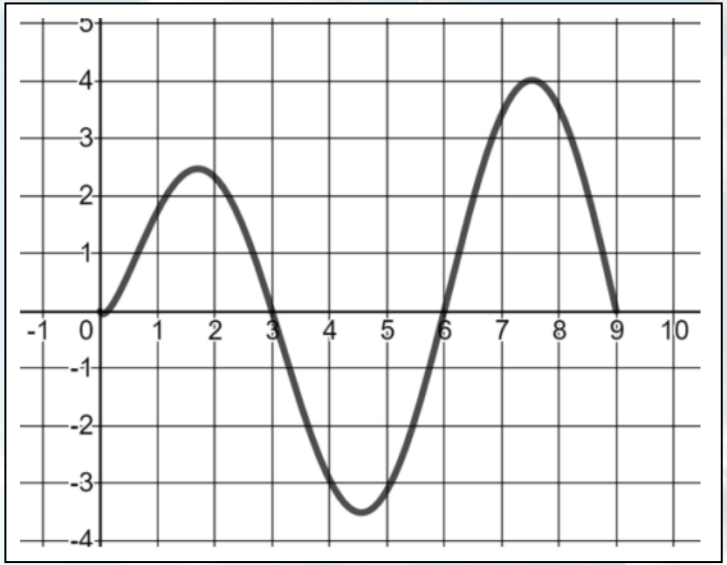


The function shown above gives the velocity of an object, $v(t)$.

Which of the following does NOT represent the displacement of the object?

- ○ $\int_0^9 v(t)dt$
- ○ $\int_0^3 v(t)dt - \int_3^6 v(t)dt + \int_6^9 v(t)dt$
- ○ $\left|\int_0^9 v(t)dt\right|$
- ○ They all represent the displacement of the object.

Participant Poll



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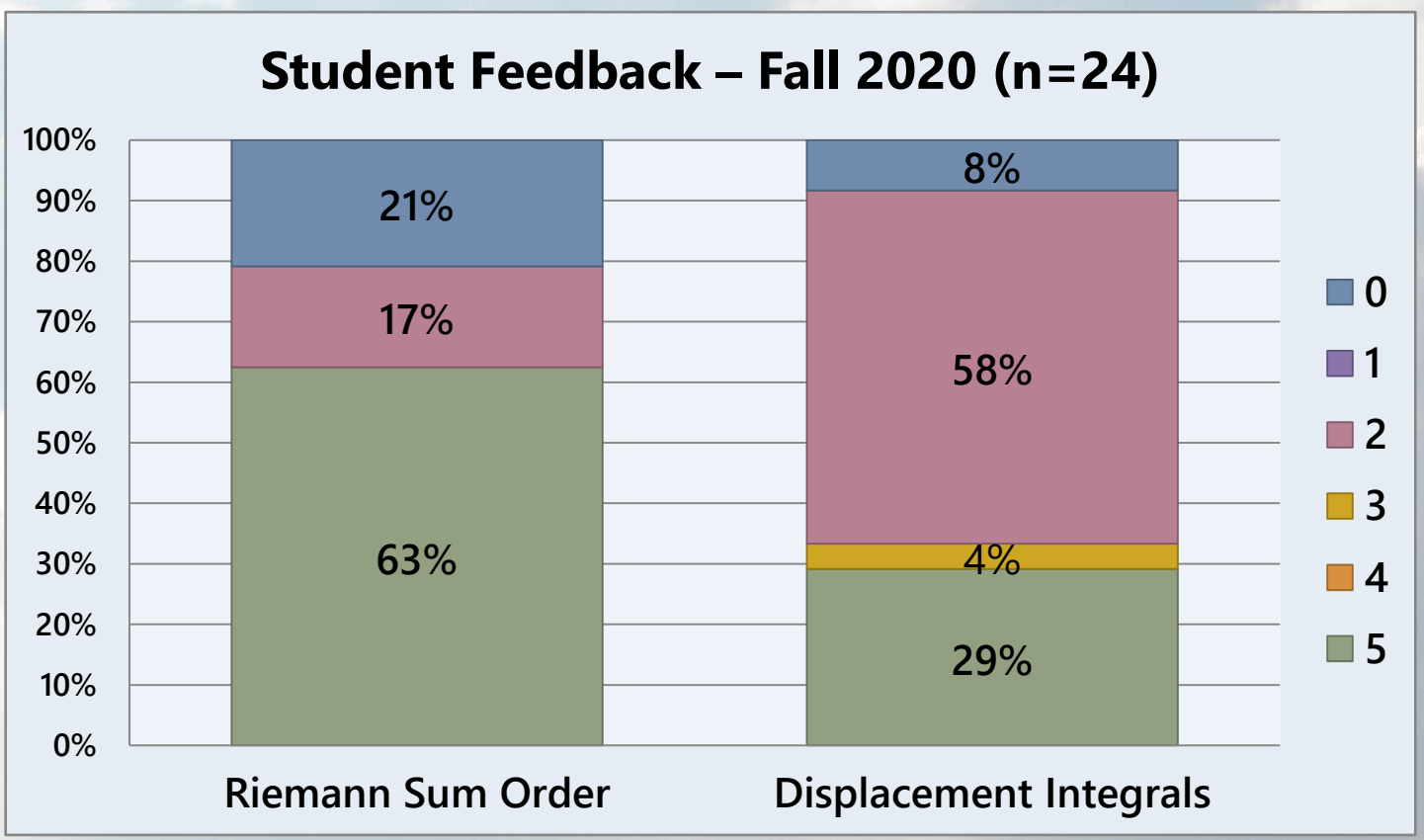
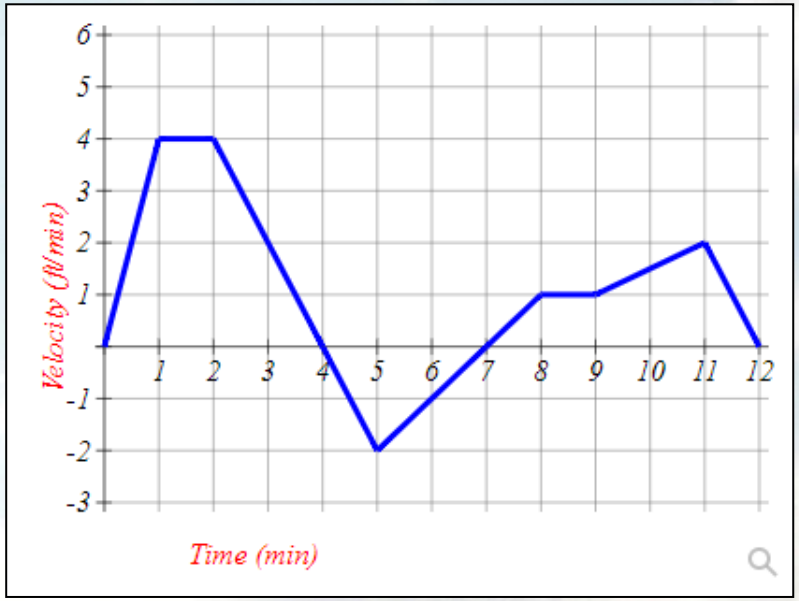
- They all represent the displacement of the object.

Answer: “B”

Student Responses



Other Types of Questions



Assume the graph above is the velocity, $v(t)$, of someone going the wrong way on a moving walkway while distracted by their cell phone (units in ft/min).

a) What is the displacement of the person during the first 12 minutes?

b) What is the total distance traveled during the first 12 minutes?

c) During which 1 minute interval does the person gain the most distance?

d) At what time during the first 9 minutes will the person be furthest from their starting position when $t = 0$?

$t =$

e) What is the average velocity of the person during the 12 minute walk?

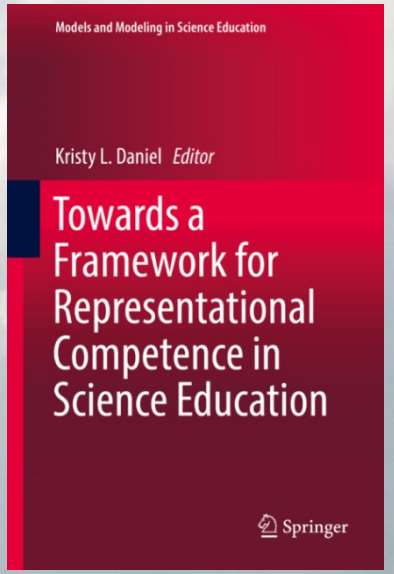
ft/min

Underlying Idea - Representational Competence (RC)

*Ability to use different **representations** of a concept as appropriate for learning, problem solving and communication.* [Kozma and Russel, 1997]

Representations = pictorial, diagram, narrative, numeric, symbolic, concrete

- Expertise = content knowledge + representational competence
- RC is key to knowledge transfer across contexts. [Steiff, 2016]
- Novice learners limit use of representations to the specific contexts in which they were introduced.
- Novice learners focus on superficial features instead of underlying meaning.



NSF Project Goals and Research Questions

Goal 1: Develop models and associated learning activities that help students develop RC in Statics and Integral Calculus.

Goal 2: Assess the effectiveness of the models and activities toward improving RC in the context of traditional coursework.

Goal 3: Identify the characteristics of modeling activities that make them effective for all learners and/or subgroups of learners.

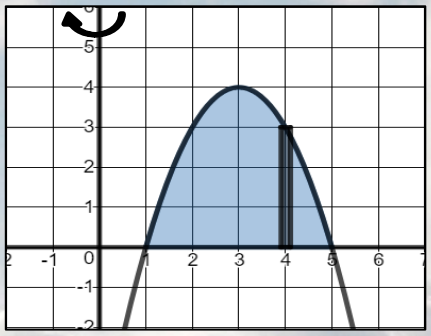
Representations in Integral Calculus

A region, *R*, bounded by the parabola $y = -(x-3)^2 + 4$ and the x-axis is rotated around the y-axis. Find the volume of the resulting solid using cylindrical shells.

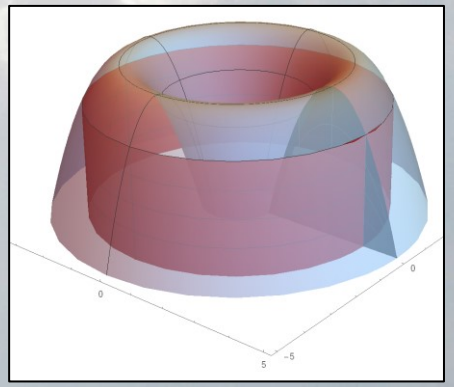
Narrative Language

$$V = \int_a^b 2\pi R h dx = \int_1^5 2\pi x(-(x-3)^2 + 4)dx$$

Symbolic/Analytical



Graphical – 2D



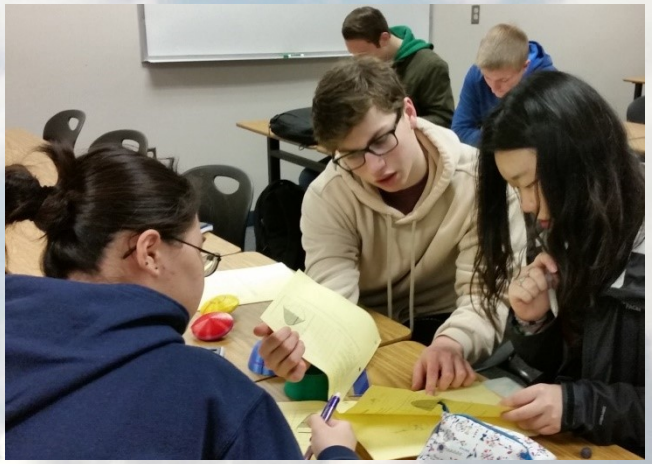
Graphical – 3D

$$V = 64\pi \approx 201.062$$

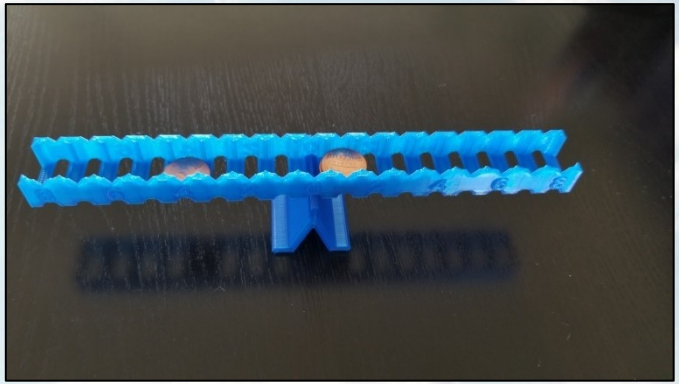
Numeric

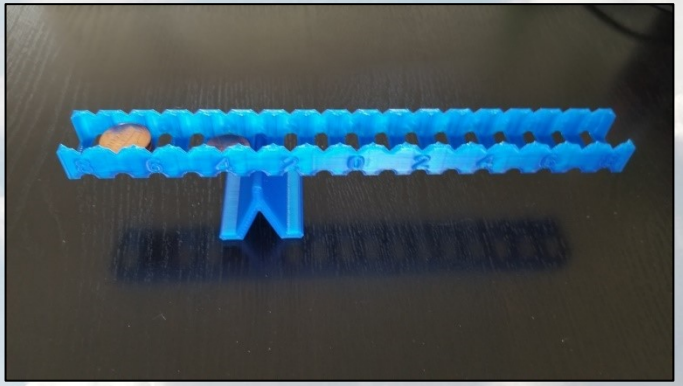
Course Goals

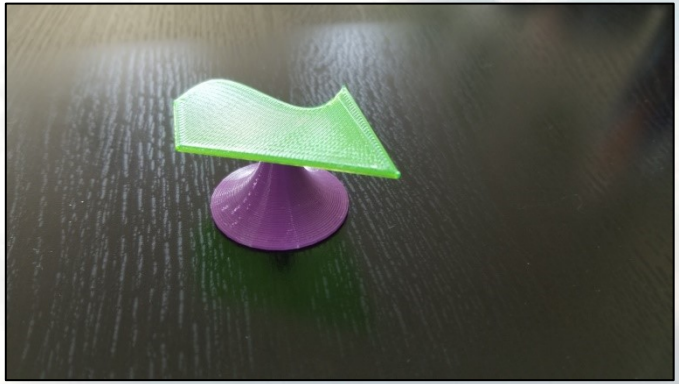
- Make classes more interactive and less lecture heavy
- Build conceptual understanding through better communication, conceptual clarity, and connections between different representations
- Help students develop better visualization skills
- Determine if tactile models contribute to better student performance over active learning with electronic aides alone

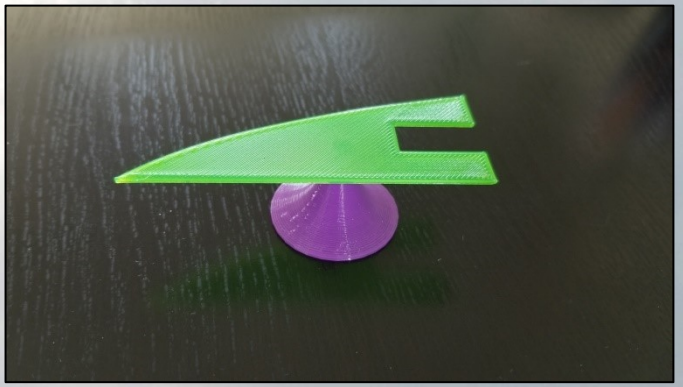


Sample Models - Centroids

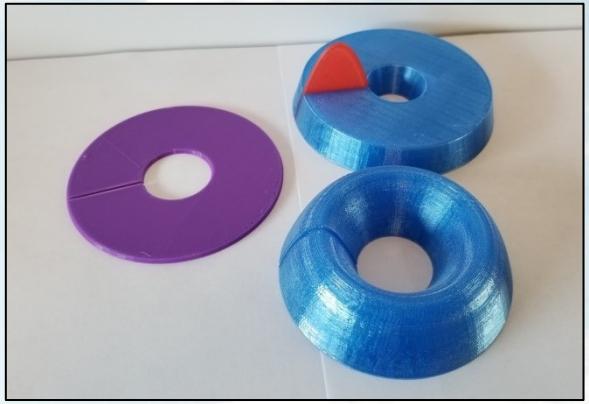


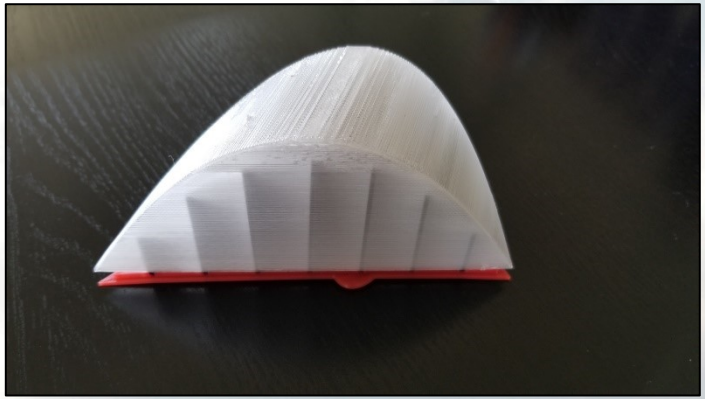


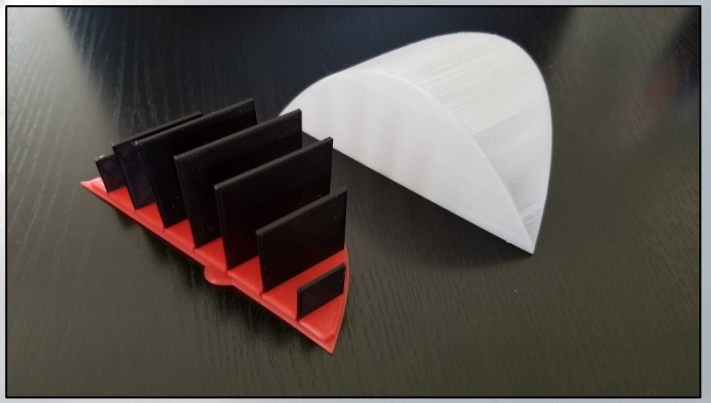




Sample Models - Volumes



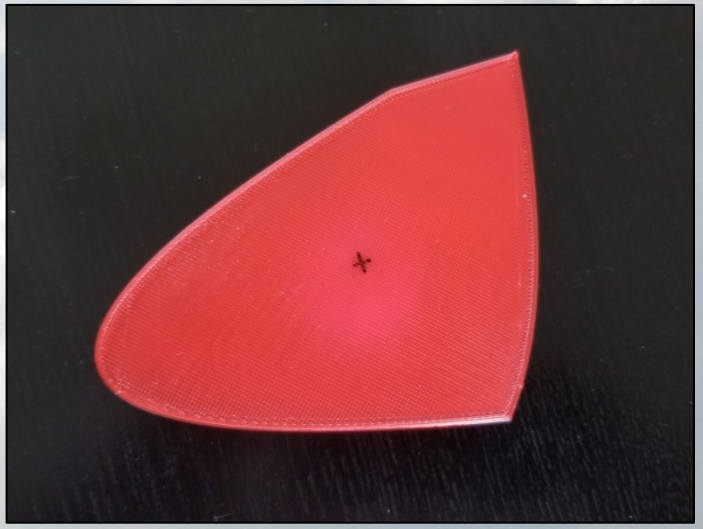




Resources

https://graspthemath.wordpress.com/integral-calculus/

- Guided notes / activity sheets.
- Links to models used (most are .stl files for 3D-printing)
- Many updates to come!



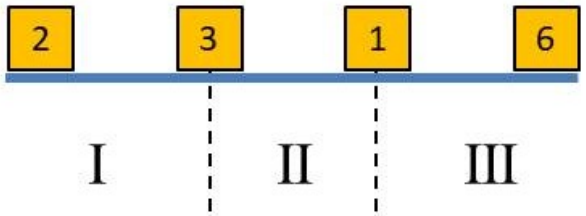
Measurement Instruments

- Concept Check Questions (Multiple Choice + Explain)
  - Students need practice with these since typical problems are procedural. Students were given a few conceptual problems as an online participation quiz, and also as part of an in-class exam.
- Student Surveys
  - Some students can have negative reactions to group work and active learning in STEM courses. We measured student reactions to the activity sheets (guided notes) and how they felt the models (or electronic aides) helped them with communication, conceptual clarity, and connections between representations.

Concept Check Questions

Centroid Pre-Test Question 1

Fall 2019



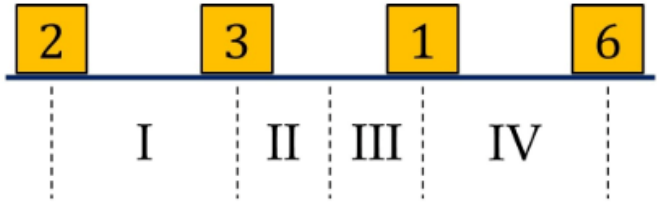
Four equally spaced weights are sitting on a bar with negligable weight.

In which region would you find the center of mass of the system?

- I
- II
- III
- Border of I and II
- Border of II and III

Please explain your answer in the box below.

Winter 2020



Four equally spaced weights are sitting on a bar with negligable weight.

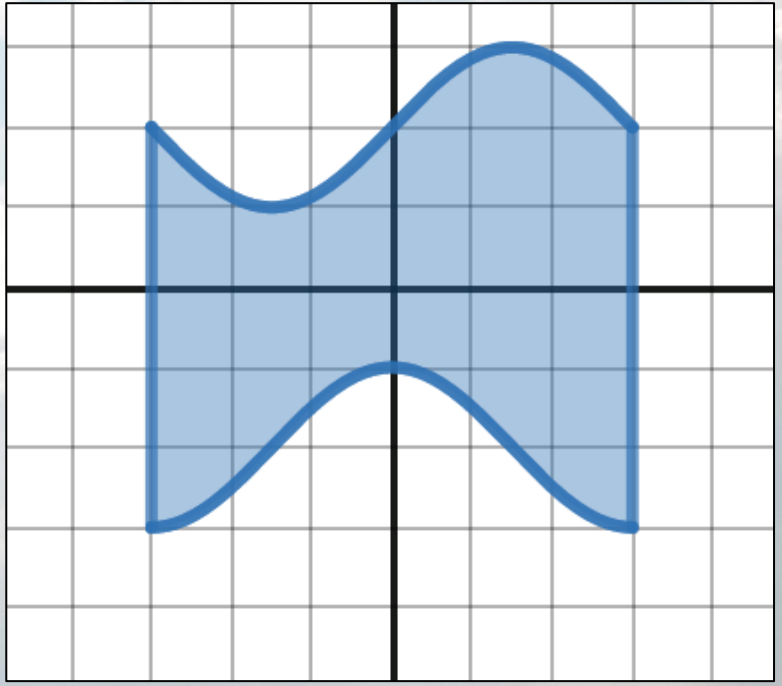
In which region would you find the center of mass of the system?

- II
- III
- IV
- Border of II and III
- Border of III and IV

Please explain your answer in the box below.

Centroid Pre-Test Question 2

A region with uniform density is shown in the figure. Where would you find the centroid of the region?



Fall 2019

- Origin
- Positive x-axis
- Negative x-axis
- Positive y-axis
- Negative y-axis
- Not on any axes

Winter 2020

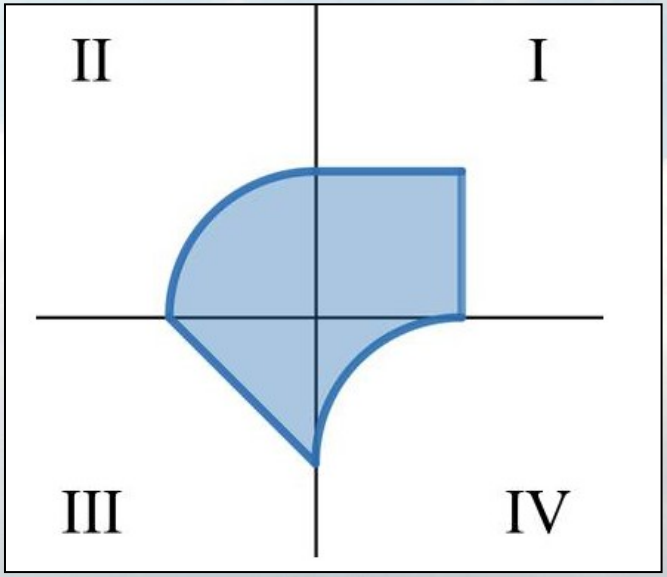
- Origin
- Positive x-axis
- Positive y-axis
- Negative y-axis
- Quadrant 1
- Quadrant 4

Centroid Pre-Test Question 3

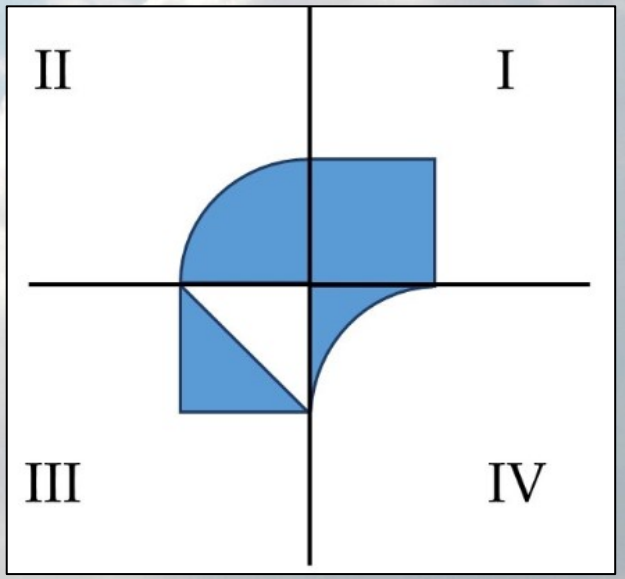
A region with uniform density is shown in the figure. In which quadrant would you find the centroid of the region?

- I
- II
- III
- IV

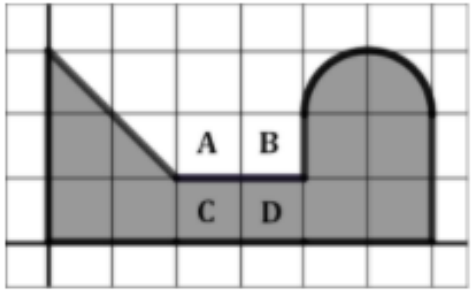
Fall 2019



Winter 2020



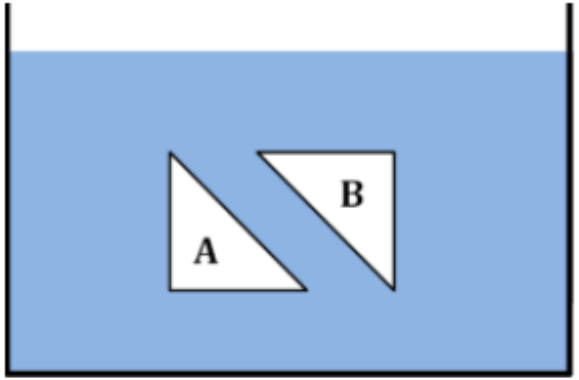
Exam 2 Questions



A region with uniform density is shown in the figure above.

Which box would contain the center of mass of this region?

- ○ A
- ○ B
- ○ C
- ○ D



Window A and B shown above are located underwater and subject to hydrostatic forces. Which of the following is TRUE?

- ○ The force on window A is greater.
- ○ The force on window B is greater.
- ○ The force is the same on A and B.

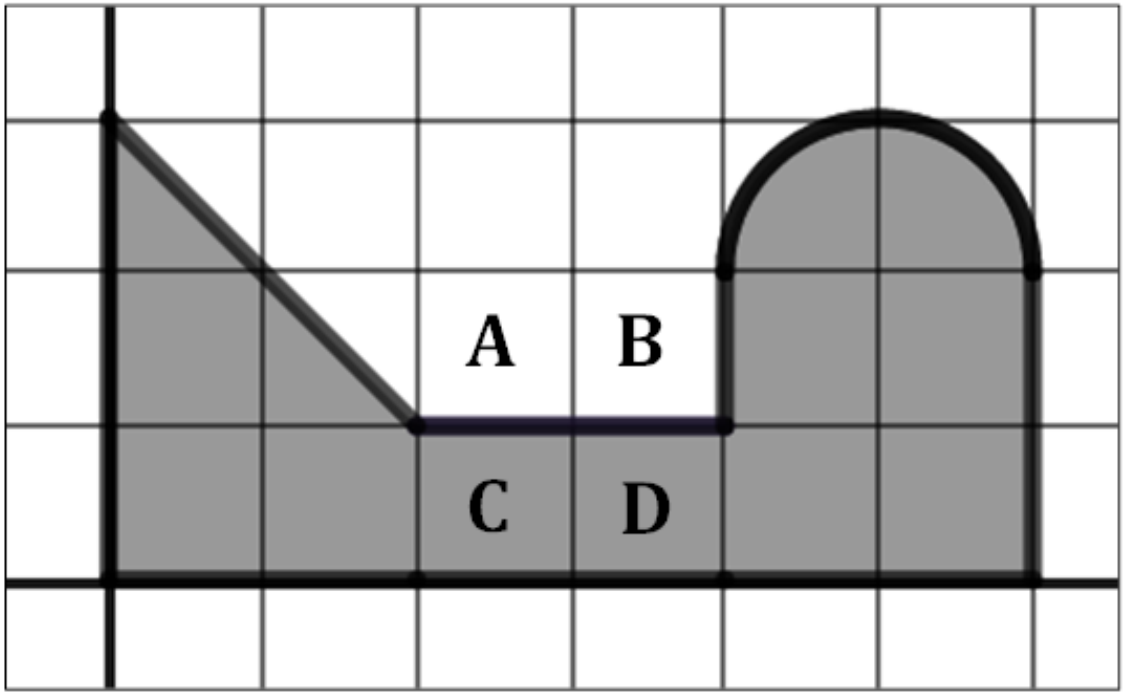
For the following system shown with weights A, B, and C, assume it balances when:

- B is double the weight of C,
- C is halfway between B and the fulcrum,
- Both A and B are the same distance from the fulcrum (but on opposite ends).

If you remove C from the balance beam, where would you place A to make the system balance?

- ○ $\frac{4}{5}$ of its current distance from the fulcrum.
- ○ $\frac{3}{4}$ of its current distance from the fulcrum.
- ○ $\frac{2}{3}$ of its current distance from the fulcrum.
- ○ $\frac{1}{2}$ of its current distance from the fulcrum.
- ○ It does not need to be moved.

Participant Poll



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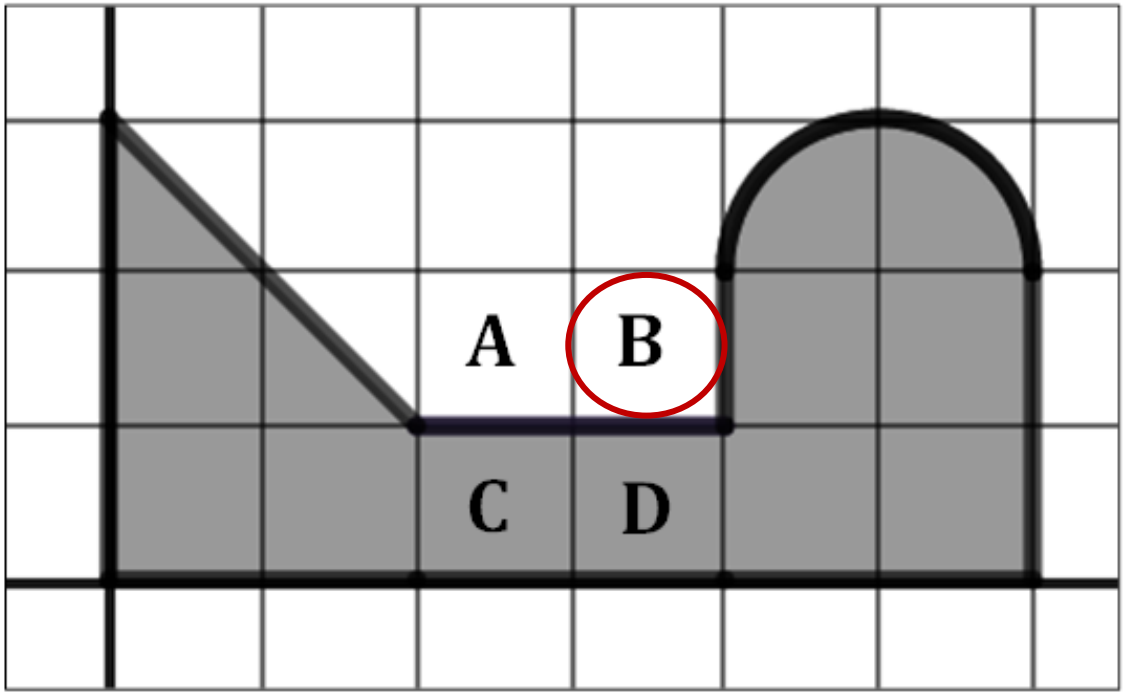
Correct Answer:

B



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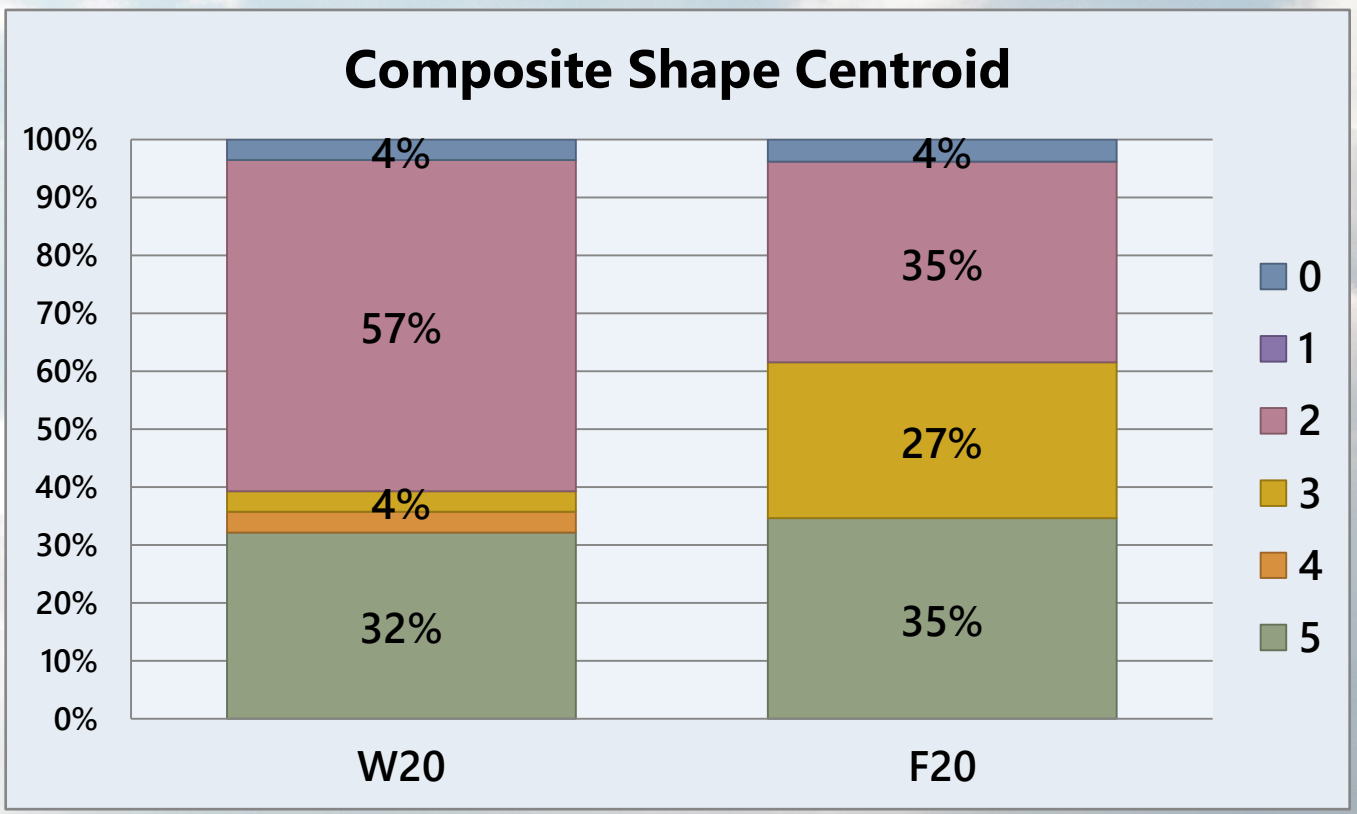
Student Results



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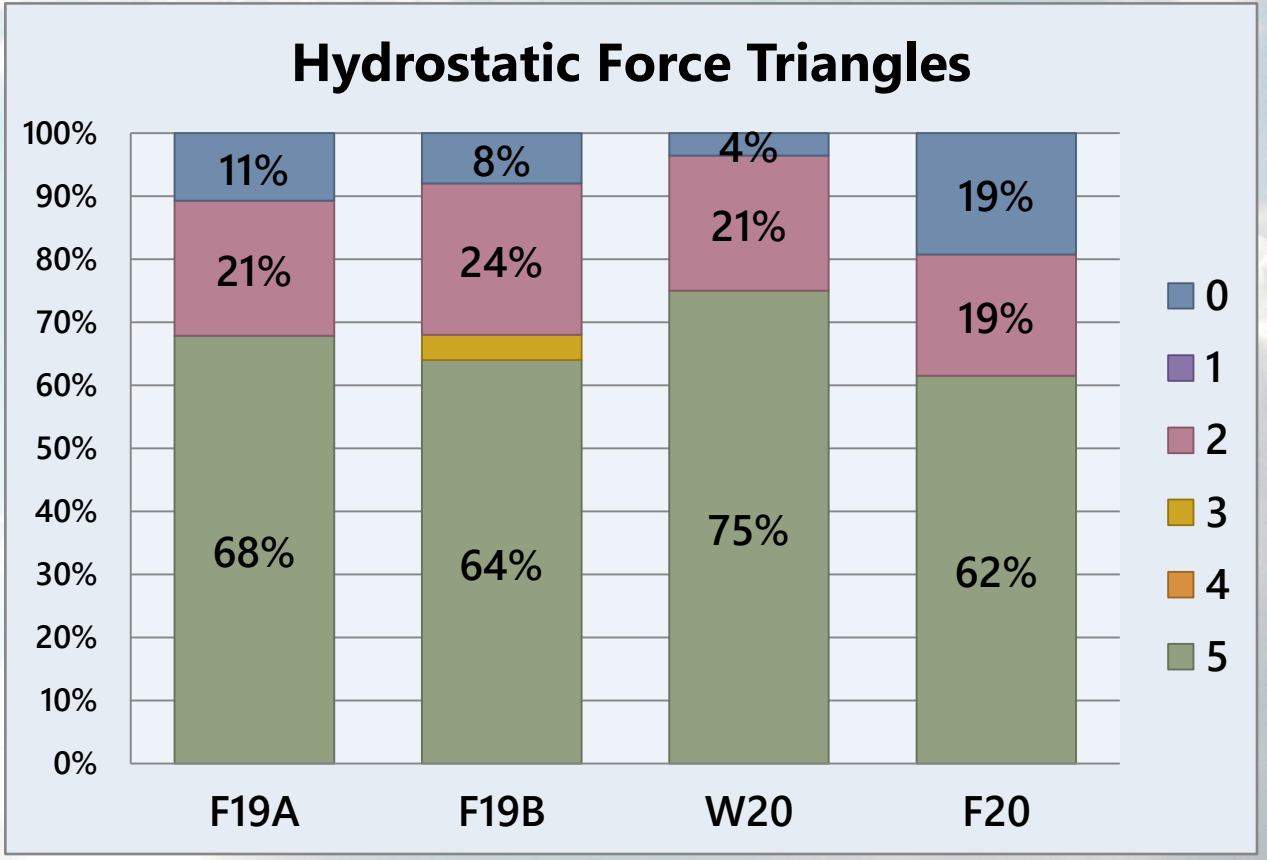

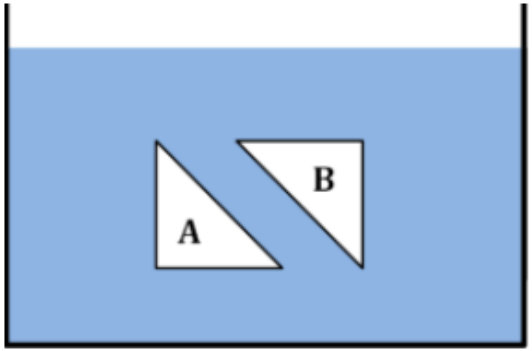

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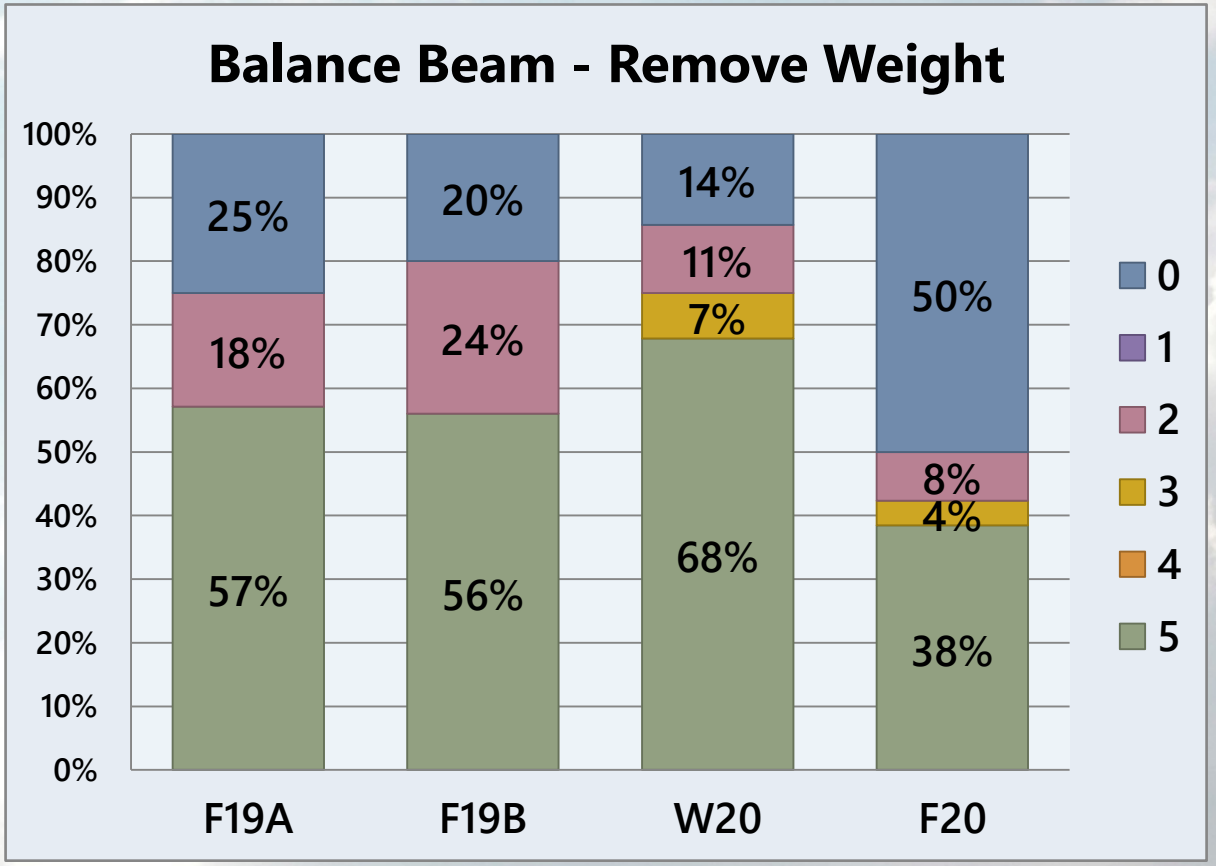

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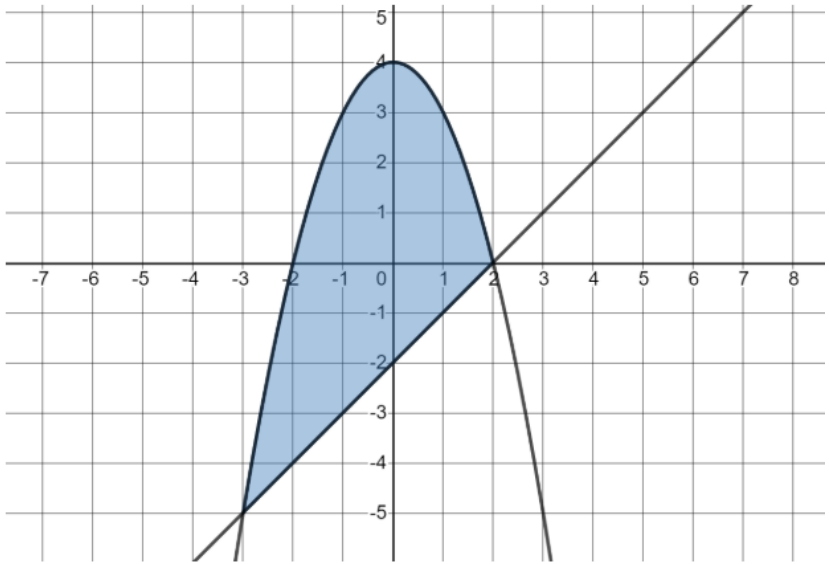
- ○ $\frac{1}{2}$ of its current distance from the fulcrum.
- ○ It does not need to be moved.

Concept Check Questions: Volumes CC1

A region R, is bounded by the functions $f(x) = 4 - x^2$ and $g(x) = x - 2$ as shown in the graph above.
The region R is rotated around the following lines to create a solid of revolution.

A. rotate around vertical line: $x = 2$
B. rotate around horizontal line: $y = -5$
C. rotate around horizontal line: $y = 4$



Put the volume of the solids in order from smallest to largest:

- ○ $A < B < C$
- ○ $A < C < B$
- ○ $B < A < C$
- ○ $B < C < A$
- ○ $C < A < B$
- ○ $C < B < A$

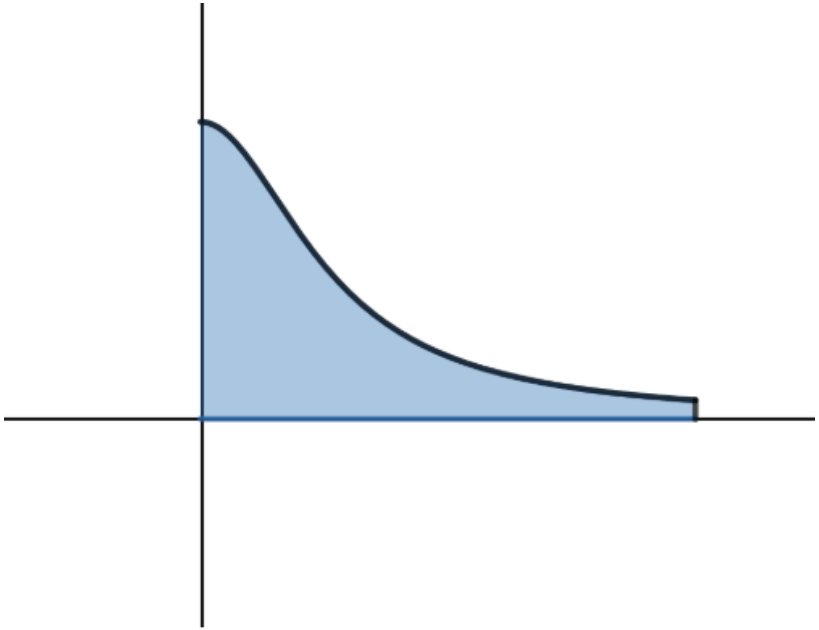
Please explain your answer in the box below.

Concept Check Questions: Volumes CC2

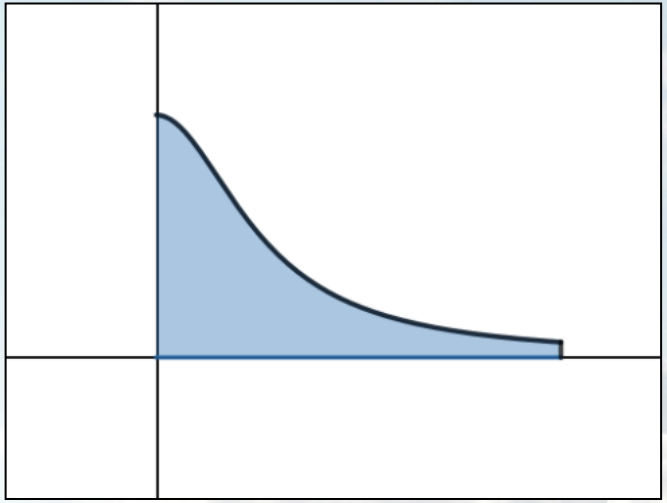
The region R shown in the graph above is rotated around the y-axis to generate a solid of revolution. A student decides to start finding the volume by dividing the region using vertical rectangles. Which method is being used to find the volume of the solid?

- ○ Disc Method, integrating with respect to x
- ○ Disc Method, integrating with respect to y
- ○ Shell Method, integrating with respect to x
- ○ Shell Method, integrating with respect to y

Please explain your answer in the box below.



Concept Check Questions: Volumes CC2



A base region R is shown in the graph above. A solid with similar cross-sections is made using the base region in the following ways:

A. Cross-sections perpendicular to the x-axis are rectangles with height equal to half the length.
B. Cross-sections perpendicular to the x-axis are equilateral triangles.
C. Cross-sections perpendicular to the x-axis are semicircles.

Put the volume of the solids in order from smallest to largest:

- ○ $A < B < C$
- ○ $A < C < B$
- ○ $B < A < C$
- ○ $B < C < A$
- ○ $C < A < B$
- ○ $C < B < A$

Exam Concept Questions: Volumes EQ1

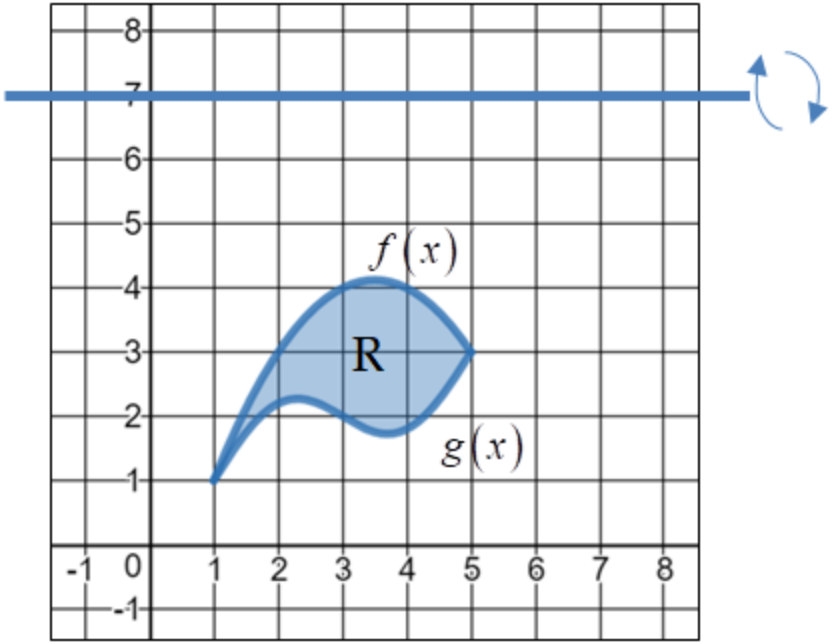
(1 pt) The region, R, is rotated around the line $y = 7$. Which integral would represent the volume of the resulting solid?

a) $\pi\int_1^5 \left(7+f(x)\right)^2 - \left(7+g(x)\right)^2 dx$

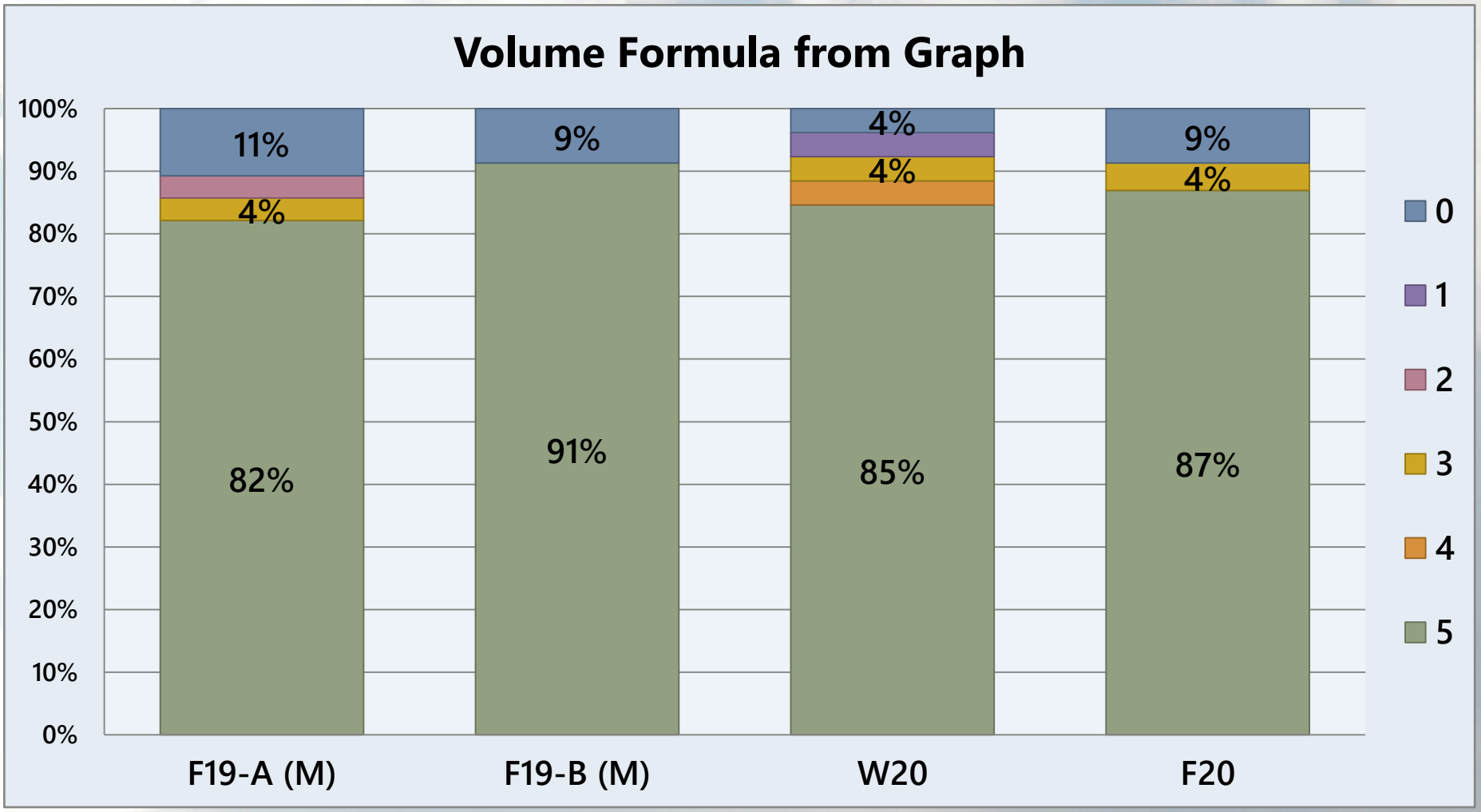
b) $\pi\int_1^5 \left(7-g(x)\right)^2 - \left(7-f(x)\right)^2 dx$

c) $2\pi\int_1^5 \left(7-f(x)\right)\left(f(x)-g(x)\right) dx$

d) $2\pi\int_1^5 \left(7-x\right)\left(f(x)-g(x)\right) dx$



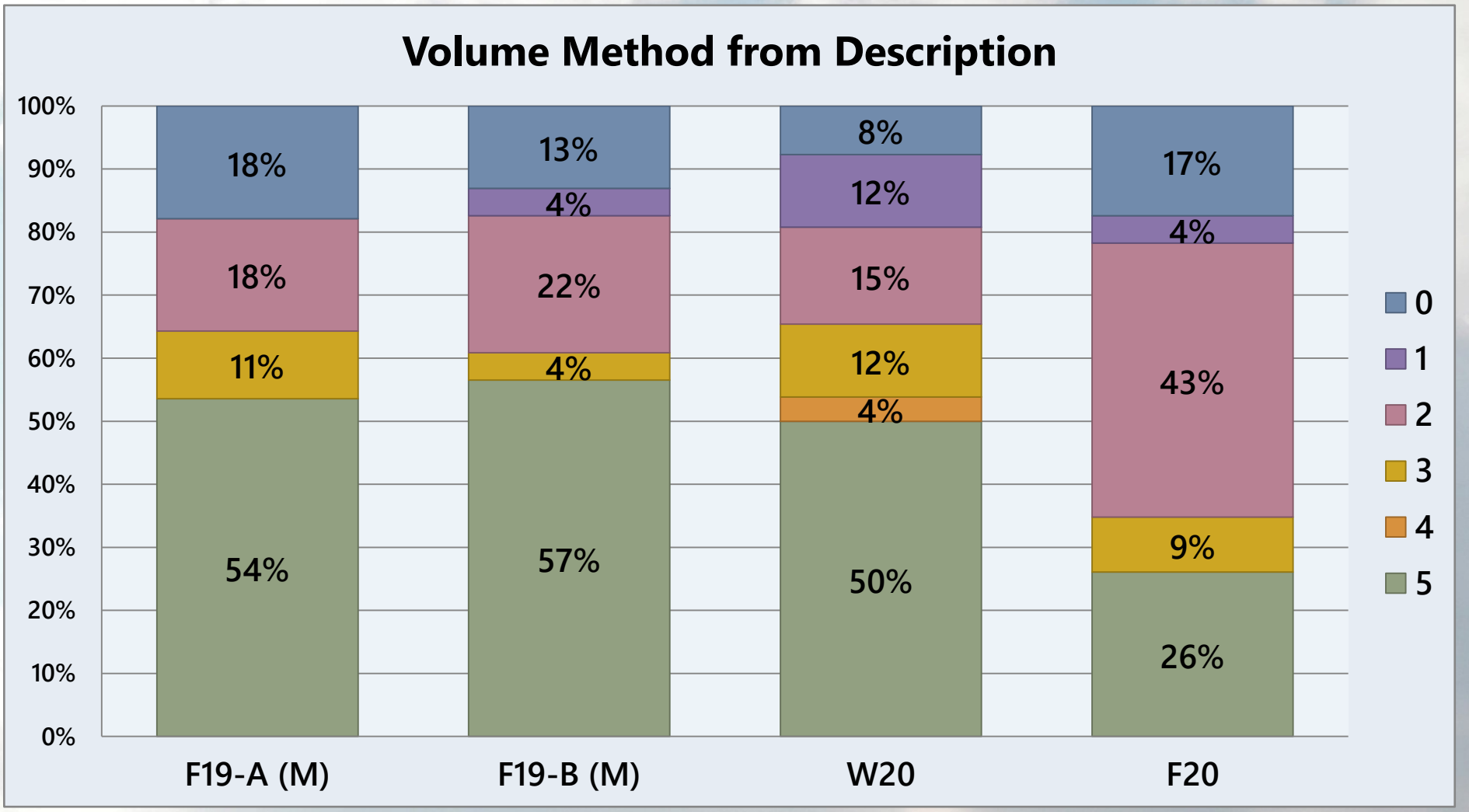
(5 pts) Explain Your Reasoning:


Volume Formula from Graph
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
11%
4%
82%
9%
91%
4%
4%
4%
85%
9%
4%
87%
F19-A (M)
F19-B (M)
W20
F20
0
1
2
3
4
5

Exam Concept Questions: Volumes EQ2

(1 pt) A region, R, lies in Quadrant I with a single point touching the y-axis. R is then rotated around the y-axis to generate a solid. If a student uses horizontal strips, which of the following are they using to find the volume of the solid?

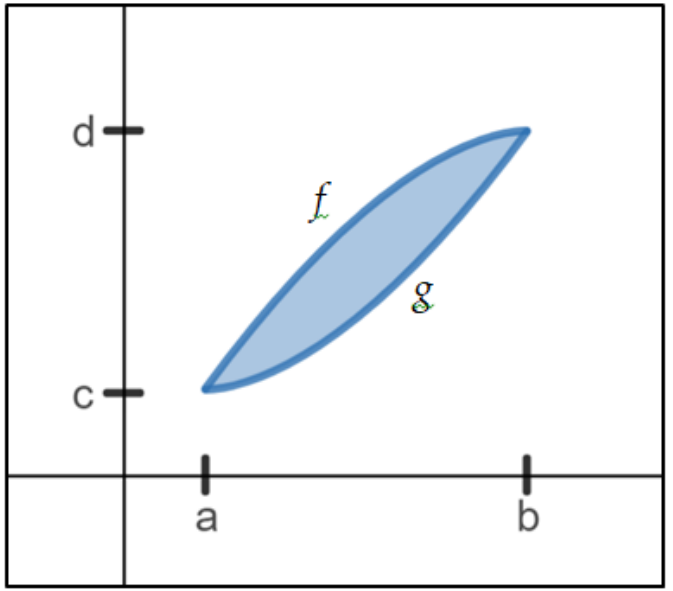
a) Discs, integrating with respect to x
b) Discs, integrating with respect to y
c) Washers, integrating with respect to x
d) Washers, integrating with respect to y
e) Cylindrical Shells, integrating with respect to x
f) Cylindrical Shells, integrating with respect to y


Volume Method from Description
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
18%
18%
11%
54%
13%
4%
22%
4%
57%
8%
12%
15%
12%
4%
50%
17%
4%
43%
9%
26%
F19-A (M)
F19-B (M)
W20
F20
0
1
2
3
4
5

Exam Concept Questions: Volumes EQ3

(1 pt) The following region is rotated around a line and a student computes the volume of the resulting solid using the following integral:

$$\int_c^d 2\pi(y-m)(g(y)-f(y))\,dy \ .$$



Which of the following best describes the axis of rotation?

a) Vertical line $x = m$, left of a

b) Vertical line $x = m$, right of b

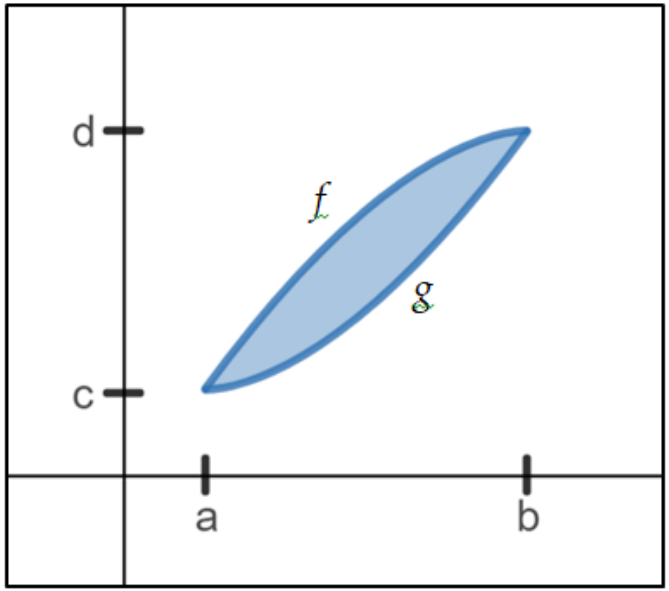
c) Horizontal line $y = m$, below c

d) Horizontal line $y = m$, above d

Participant Poll: Volumes EQ3

(1 pt) The following region is rotated around a line and a student computes the volume of the resulting solid using the following integral:

$$\int_c^d 2\pi(y-m)(g(y)-f(y))\,dy \ .$$



Which of the following best describes the axis of rotation?

a) Vertical line $x = m$, left of a

b) Vertical line $x = m$, right of b

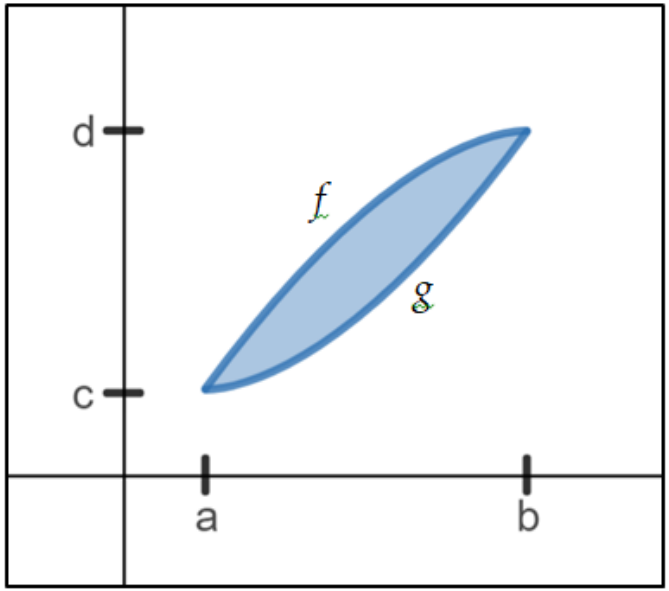
c) Horizontal line $y = m$, below c

d) Horizontal line $y = m$, above d

Participant Poll: Volumes EQ3

(1 pt) The following region is rotated around a line and a student computes the volume of the resulting solid using the following integral:

$$\int_c^d 2\pi(y-m)(g(y)-f(y))\,dy \ .$$



Which of the following best describes the axis of rotation?

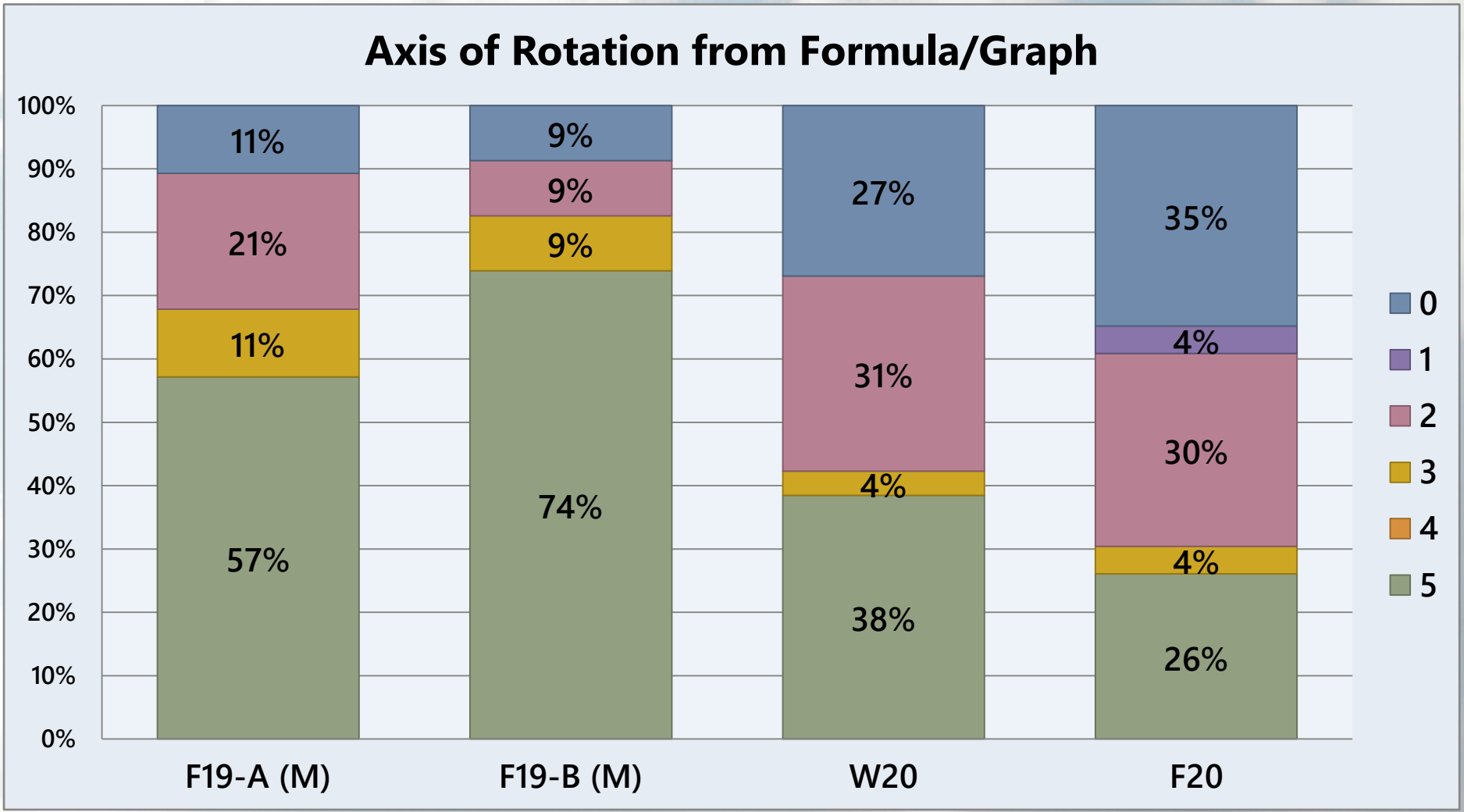
a) Vertical line $x = m$, left of a

b) Vertical line $x = m$, right of b

c) Horizontal line $y = m$, below c

d) Horizontal line $y = m$, above d

Correct Answer: C


Axis of Rotation from Formula/Graph
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
11%
21%
11%
57%
9%
9%
9%
74%
27%
31%
4%
38%
35%
4%
30%
4%
26%
F19-A (M)
F19-B (M)
W20
F20
0
1
2
3
4
5

Exam Concept Questions: Volumes EQ4

(1 pt) A solid is made by rotating the region bounded inside the unit circle about the line $x = 3$. A student decides to use cylindrical shells to compute the volume. What should be used for the value of a shell radius?

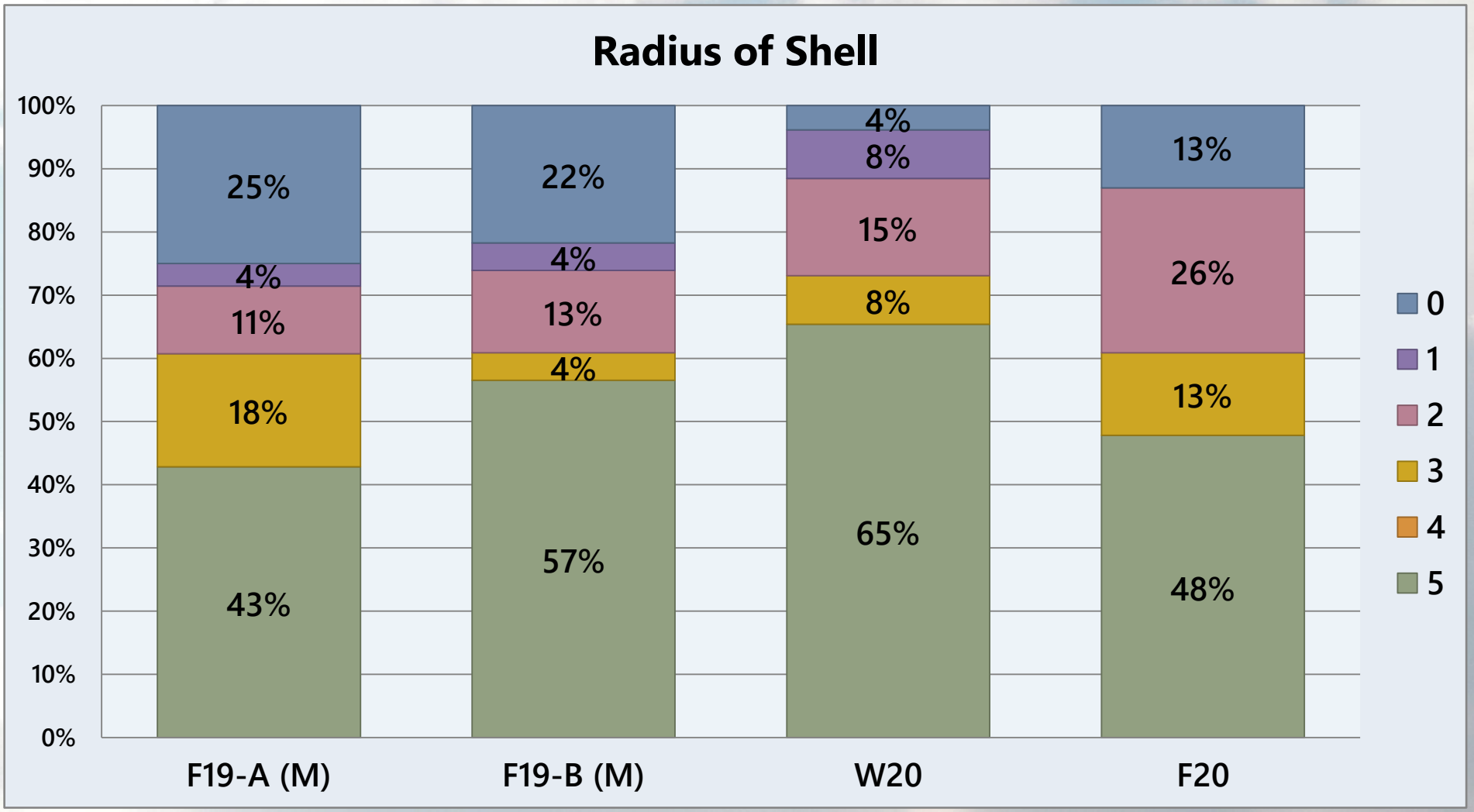
a) x

b) $x - 3$

c) $3 - x$

d) $\sqrt{1 - x^2} - 3$

e) $3 - \sqrt{1 - x^2}$

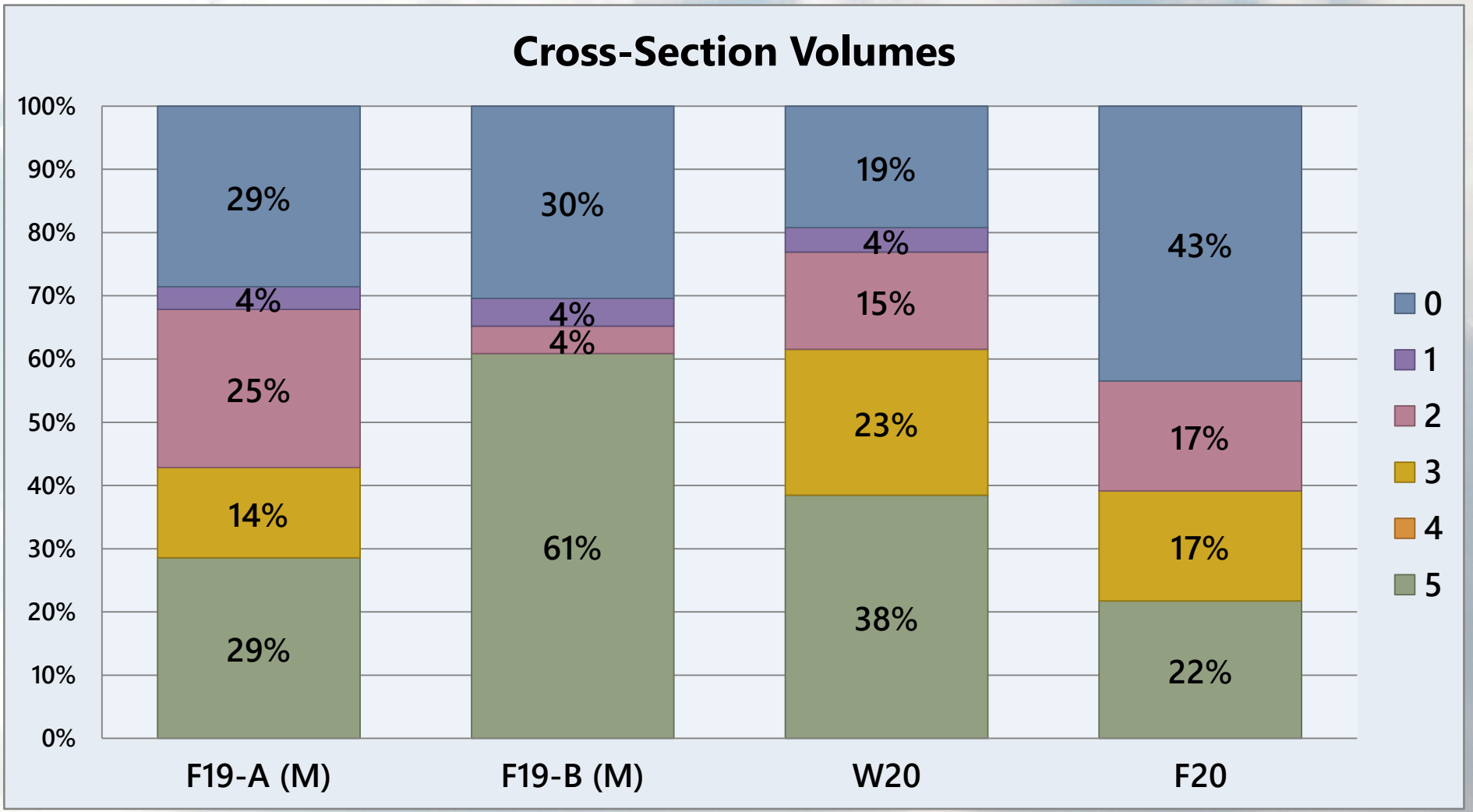

Radius of Shell
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
25%
4%
11%
18%
43%
22%
4%
13%
4%
57%
4%
8%
15%
8%
65%
13%
26%
13%
48%
F19-A (M)
F19-B (M)
W20
F20
0
1
2
3
4
5

Exam Concept Questions: Volumes EQ5

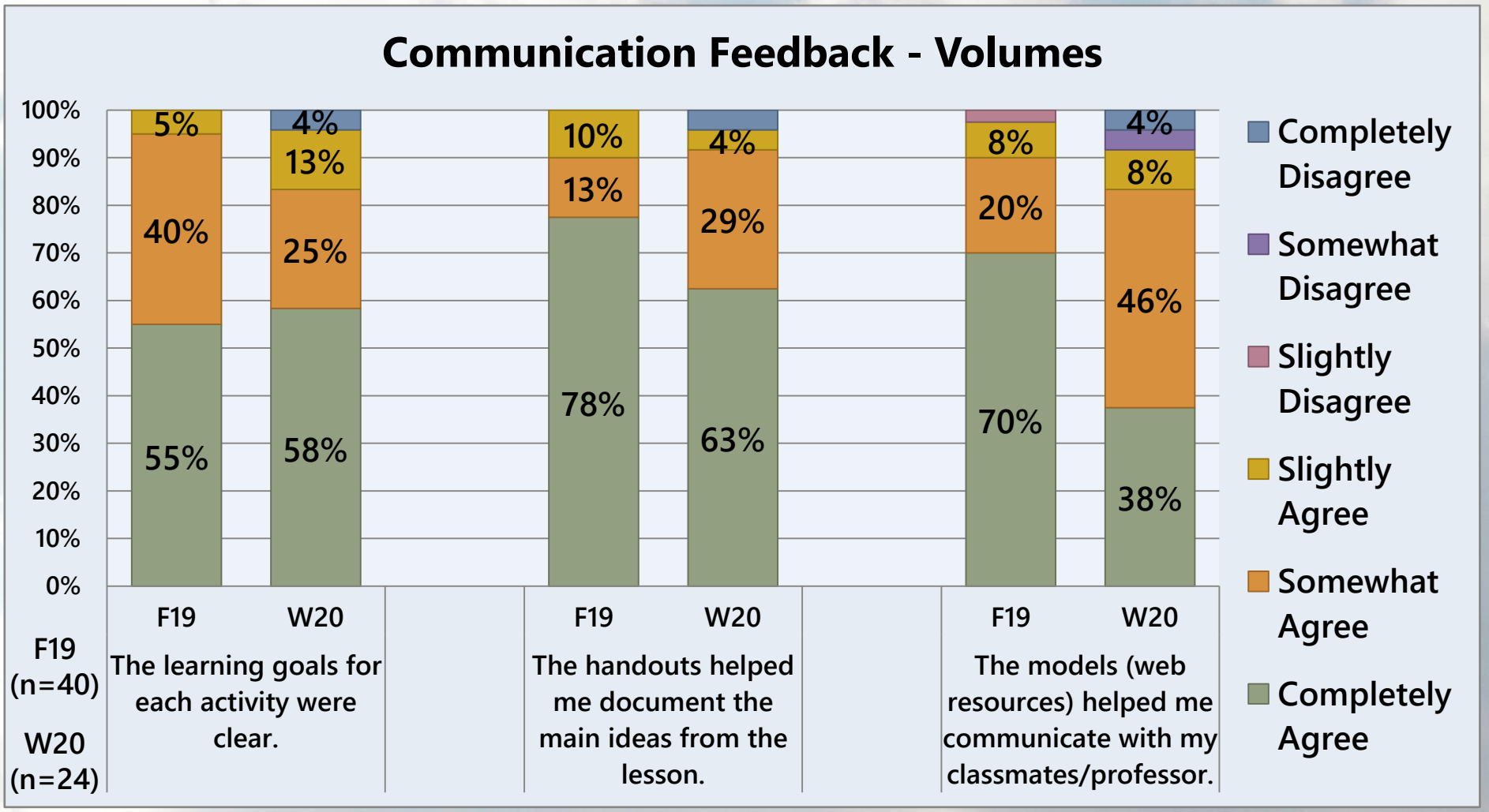
A region R is bounded by the x-axis and $y = \sin(x)$ on the interval $[0, \pi]$. Four solids are described as having a base region R, where each cross-section perpendicular to the x-axis is shown in the choices. Which cross-section shape results in a solid that DOES NOT have the same volume as the others?

(Note: Figures are given to help interpret unfamiliar geometric terms. They are not drawn to scale relative to one another)

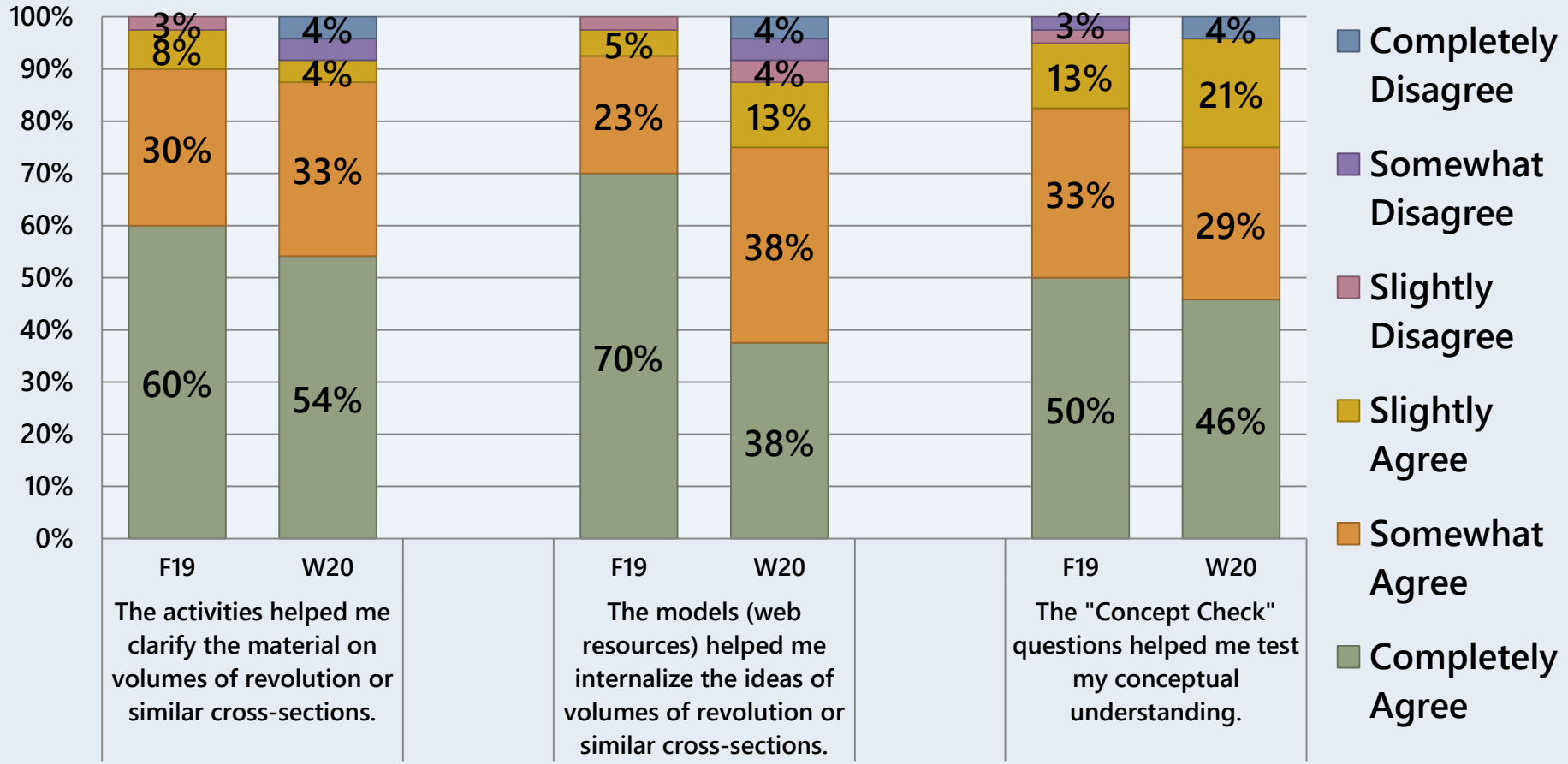
- ○ Rectangle with height half the base 
- ○ Obtuse Triangle with height equal to the base 
- ○ Isosceles Right Triangle with Leg as the base 
- ○ Isosceles Right Triangle with Hypotenuse as the base 
- ○ They all have the same volume


Cross-Section Volumes
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
29%
4%
25%
14%
29%
30%
4%
4%
61%
19%
4%
15%
23%
38%
43%
17%
17%
22%
F19-A (M)
F19-B (M)
W20
F20
0
1
2
3
4
5

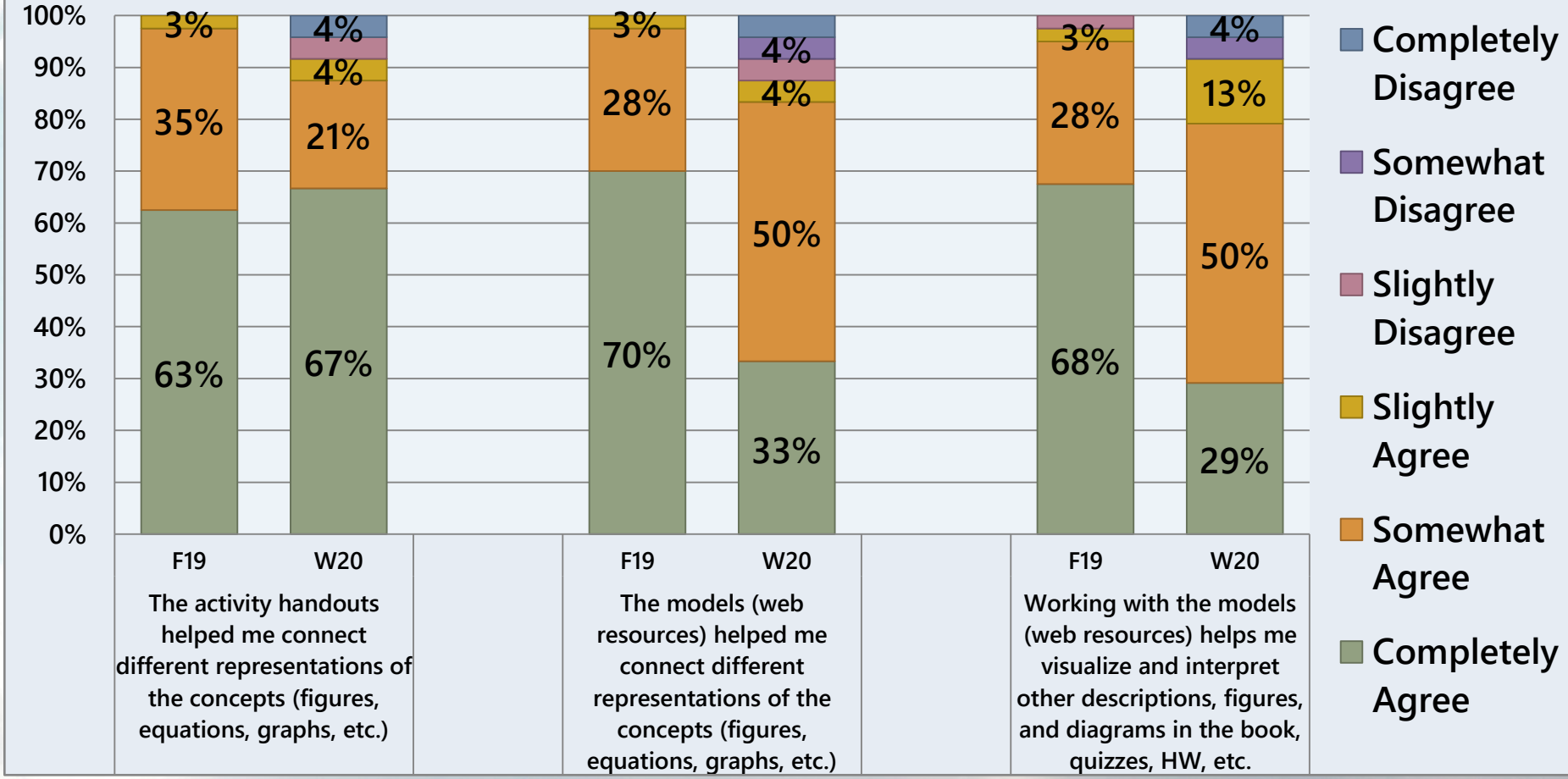
Student Feedback Results


Communication Feedback - Volumes
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
5%
40%
55%
4%
13%
25%
58%
10%
13%
78%
4%
29%
63%
8%
20%
70%
4%
8%
46%
38%
F19
W20
F19
W20
F19
W20
F19 (n=40)
W20 (n=24)
The learning goals for each activity were clear.
The handouts helped me document the main ideas from the lesson.
The models (web resources) helped me communicate with my classmates/professor.
Completely Disagree
Somewhat Disagree
Slightly Disagree
Slightly Agree
Somewhat Agree
Completely Agree

Conceptual Clarity Feedback - Volumes



Connections Feedback - Volumes



Sample Student Comments on Models (Fall 2019)

- "The models were brilliant! I couldn't have conceptualized this material without them."
- "I felt the 3D models were very helpful especially when it came to showing the difference between discs and washers. I like how the slice could come out and how it matched the equation on the paper. It really helped show how it would rotate about an axis."
- "The models were super helpful for this section. The models really helped me understand the 3 dimensions of the rotations with shells and washers etc."
- "The models really helped me visualize what we were doing and how to use the formulas to solve the problems at hand."

Conclusions

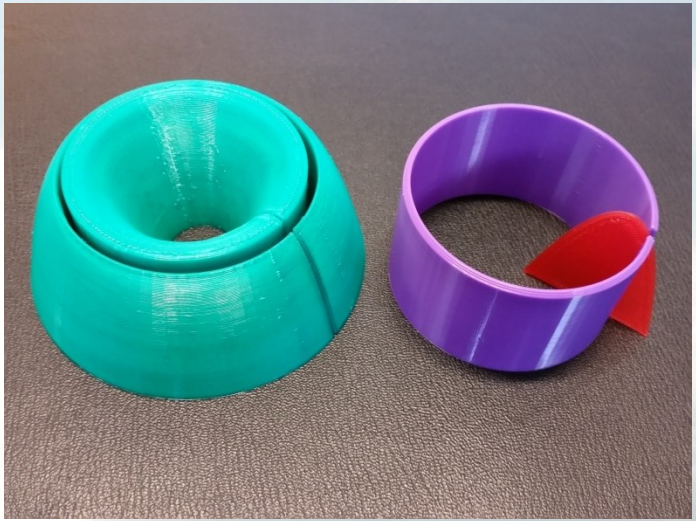
- Guided notes / activity sheets with models are an effective way of engaging students with calculus material. Student reactions to the intervention methods are extremely positive.
- Students generally find physical models more helpful than web resources for internalizing ideas and forming connections between representations.
- Conceptual Questions can help students think about material in a more conceptual way rather than just rote procedures. They are also a great way of assessing students on their conceptual understanding and not just procedural prowess.


What's
NEXT

ONLINE REVAMPING!!!

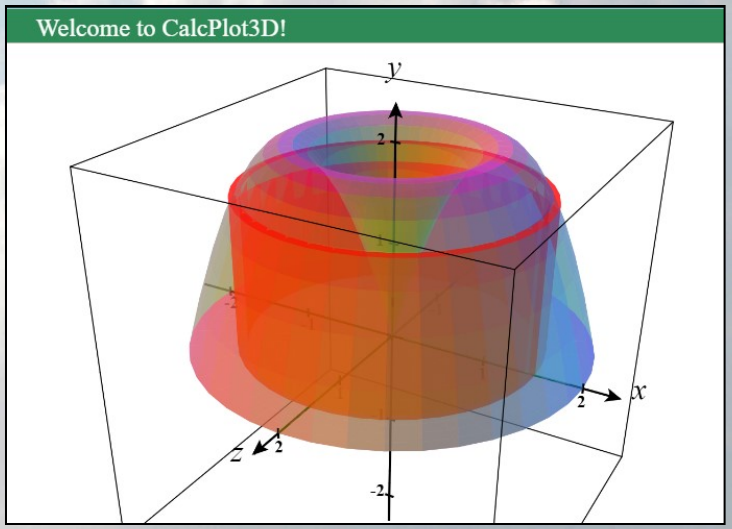
Intervention Sections

- Guided Notes / Activity Sheets
- Physical Models (Personal Sets)



Control Sections

- Guided Notes / Activity Sheets
- Electronic Models (Web-Based)



Next Steps

- Individual Student Kits are being developed for use in Winter and Spring quarter in 2021. Students should be able to use the kits even if courses are in an online format.

- Data will collected from several instructors at a select Washington community colleges during Winter and Spring 2021.

- Revised models and worksheets will soon be added to the website in the near future.

https://graspthemath.wordpress.com/integral-calculus/

Acknowledgements

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